Met 205A, Assignment 3,

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Question 1: From the primitive equations (6a) & (6b), derive an expression for vorticity.

Solution: Start with the primitive equations in log-pressure coordinates:

$$(1.\text{a})\quad \frac{Du}{Dt} - fv + \Phi_x = X \qquad\qquad (2.\text{a})\quad \frac{Dv}{Dt} + fu + \Phi_y = Y$$

Expand the total derivative term (1st term) in $(1.\text{a})$ & $(2.\text{a})$ using definition of total derivative:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

Substitute expanded form of the total derivative into $(1.\text{a})$ & $(2.\text{a})$:

$$(1.\text{b})\quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \Phi_x = X$$

$$(2.\text{b})\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \Phi_y = Y$$

Take $\frac{\partial}{\partial y}(1.\text{b})$:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \Phi_x\right) = \frac{\partial X}{\partial y} \Rightarrow$$

$$(1.\text{c})\quad \frac{\partial}{\partial y}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial^2 y} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y\partial z} - \frac{\partial}{\partial y}(fv) + \frac{\partial}{\partial y}\Phi_x = \frac{\partial X}{\partial y}$$

Take $\frac{\partial}{\partial x}(2.\text{b})$:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \Phi_y\right) = \frac{\partial Y}{\partial x} \Rightarrow$$

$$(2.\text{c})\quad \frac{\partial}{\partial x}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial^2 x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial x\partial z} + \frac{\partial}{\partial x}(fu) + \frac{\partial \Phi_y}{\partial x} = \frac{\partial Y}{\partial x}$$

Rearrange partial derivatives of $(2.\text{c})$. $f = 2\Omega\sin\phi$, where ϕ = latitude. Thus, f does not vary in the x-direction, and $\therefore$, $\partial f/\partial x = 0$

$$(2.\text{d})\quad \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial^2 x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial x\partial z} + f\frac{\partial u}{\partial x} + \cancel{u\frac{\partial f}{\partial x}} + \frac{\partial \Phi_y}{\partial x} = \frac{\partial Y}{\partial x}$$

Rearrange partial derivatives of $(1.\text{c})$

$$(1.\text{d})\quad \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial^2 y} + \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y\partial z} - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y} + \frac{\partial \Phi_x}{\partial y} = \frac{\partial X}{\partial y}$$

Subtract $(1.\text{d})$ from $(2.\text{d})$ producing:

$$(3.\text{a})\quad \frac{\partial}{\partial t}\frac{\partial v}{\partial x} - \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 v}{\partial^2 x} - u\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y} - v\frac{\partial^2 u}{\partial^2 y} +$$

$$\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 v}{\partial x\partial z} - w\frac{\partial^2 u}{\partial y\partial z} + f\frac{\partial u}{\partial x} + f\frac{\partial v}{\partial y} + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Recognizing that $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, rearrange (3.a) to group vorticity terms accordingly.

$$(3.\text{b}) \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) +$$
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Rewrite (3.b) by substituting ζ for all $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ terms.

$$(3.\text{c}) \frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) +$$
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Recognizing that the 1st four terms on the LHS are the expanded form of the total derivative of vorticity (Eulerian term + 3 advection terms), rewrite (3.c) by replacing the expanded terms with the total derivative term $\frac{D\zeta}{Dt}$.

$$(3.\text{d}) \frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Factor out these vorticity terms $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ and rewrite equation.

$$(3.\text{e}) \frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

Factor out divergence terms

$$(3.\text{f}) \frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

$$(3.\text{g}) \frac{D\zeta}{Dt} + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

$$(3.\text{g}) \frac{D\zeta}{Dt} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{\partial f}{\partial y} + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

The term $v\frac{\partial f}{\partial y}$ can be recognized as the $v-$component term of the advection of the Coriolis parameter, f.

Since the coriolis parameter depends only on y, one could rewrite $v\frac{\partial f}{\partial y}$ in the form of the total derivative, $\frac{Df}{Dt}$. The full definition of the total derivative of f is $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$. Since f depends only on y, the eulerian term $\frac{\partial f}{\partial t}$ and the other two advection terms $\left(u\frac{\partial f}{\partial x} + w\frac{\partial f}{\partial z}\right)$ all equal zero, since the derivative of a constant (in this case f) is zero. (Holton, p. 101).

$$(3.\text{h}) \frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{\partial \Phi_y}{\partial x} - \frac{\partial \Phi_x}{\partial y} = \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}$$

$$(3.\text{i})\quad \boxed{\frac{D}{Dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\left(\frac{\partial \Phi_x}{\partial y}-\frac{\partial \Phi_y}{\partial x}\right)+\left(\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}\right)}$$

$(3.\text{i})$ is the vorticity equation in log-pressure coordinates. The term on the LHS of the equation is the total derivative of absolute vorticity $(\text{relative + planetary})$. The four terms on the RHS are the divergence term, the tilting or twisting term, the solenoidal term, and non-conservative force terms (ie. friction).

Question 2) Explain clearly why a) vorticity and b) Ertel's potential vorticity (Eq 3/4) are useful quantities is atmospheric science, and how they are different.

Vorticity is a useful quantity in atmospheric science because it can be used to forecast weather in certain circumstances. For example, in conditions where the flow is exclusively horizontal and non-divergent, (or where one may neglect vertical motion and friction), the vorticity equation above can be reduced to a simplified form known as the barotropic vorticity equation, $\frac{D_h(\zeta_g+f)}{Dt}=0$, where ζ_g = geostrophic vorticity. Rewriting this equation in the form of a streamfunction, $\psi(x,y)$, allows for vorticity and velocity to be contained in the variable ψ. Thus, solving the barotropic vorticity equation numerically allows for the prognosis of both vorticity and velocity. This implies that weather forecasting can be achieved in limited conditions from a single quantity, vorticity.

Ertel's potential vorticity (PV) is a useful quantity in atmospheric science because it is a conserved quantity following the motion in adiabatic flow. The significance of this is that it can be used a tracer of air masses, especially when comparing tropospheric and stratospheric air masses. This is because the stratospheric air masses have consistently higher PV values (> 2) than tropospheric ones (< 1.5) . Thus, PV can be used as a fairly accurate indicator of the height of the tropopause. In addition, high PV values found below the tropopause indicate stratospheric air that has been caught in a “fold” below that tropopause.

An important application of this characteristic is the mapping of the Antarctic polar vortex, which has significantly higher values of PV than neighboring air outside of the vortex. PV is a also useful as a measure of other changing variables.