Met 205a, Homework # 3

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2) a) Derive an expression for the thermal wind relationship starting from the quasi-geostrophic equations given by Eq. 6

Start with the Primative equations on a β-plane

$$(1.a)\ \frac{Du}{Dt} - fv + \Phi_x = X \qquad (2.a)\ \frac{Dv}{Dt} + fu + \Phi_y = Y$$

After scale analysis, the coriolis force and pressure gradiant force can be seen to be the dominant terms. Thus, the horizonal components of the geeostrophic wind in log-pressure cooordinates are:

$$(1.b) u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \qquad (2.b)\ v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

Take $\frac{\partial}{\partial z}$ of u_g and v_g

$$(1.d)\frac{\partial u_g}{\partial z} = \frac{\partial}{\partial z}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) \Rightarrow \frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial z}\right)$$

$$(2.d)\frac{\partial v_g}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) \Rightarrow \frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial z}\right)$$

Differentiate geopotential with respect to z

$$\Phi_z = \frac{RT}{H} \Rightarrow \quad \frac{\partial \Phi_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{RT}{H}\right) \Rightarrow \frac{\partial \Phi_z}{\partial z} = \frac{RT}{H}$$

Substitute $\frac{\partial \Phi_z}{\partial z}$ into $(1.d)$ and $(2.d)$

$$(1.e)\ \frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{RT}{H}\right) \Rightarrow \boxed{\frac{\partial u_g}{\partial z} = -\frac{1}{f}\frac{R}{H}\frac{\partial T}{\partial y}}$$

$$(2.e)\frac{\partial v_g}{\partial z} = \frac{1}{f}\frac{\partial}{\partial x}\left(\frac{RT}{H}\right) \Rightarrow \boxed{\frac{\partial v_g}{\partial z} = \frac{R}{Hf}\frac{\partial T}{\partial x}}$$