Turbulent Kinetic Energy Equation for Turbulence (incompressible flow)

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \tag{2}$$

Multiply (2) by u_i

$$u_i \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial^2 u_i}{\partial x_j^2} \tag{3}$$

$$\frac{\partial u_i u_i}{2\partial t} + u_i \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial^2 u_i}{\partial x_j^2} \tag{4}$$

Substitute in the Reynolds decomposition,

$$u_i = \bar{u}_i + u_i' \tag{5}$$

and average

$$\begin{aligned}
&\overline{\frac{\partial u_i u_i}{2\partial t} + u_i \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_i} + \nu u_i \frac{\partial^2 u_i}{\partial x_j^2}} = \\
&\overline{\frac{\partial(\bar{u}_i + u_i')(\bar{u}_i + u_i')}{2\partial t} + (\bar{u}_i + u_i')\frac{\partial(\bar{u}_i + u_i')(\bar{u}_j + u_j')}{\partial x_j}} = \\
&-\overline{\frac{1}{\rho}(\bar{u}_i + u_i')\frac{\partial(\bar{p} + p')}{\partial x_i} + \nu(\bar{u}_i + u_i')\frac{\partial^2(\bar{u}_i + u_i')}{\partial x_j^2}}
\end{aligned} \tag{6}$$

or

$$\begin{aligned}
&\frac{\partial \bar{u}_i \bar{u}_i}{2\partial t} + \frac{\overline{\partial u_i' \bar{u}_i}}{2\partial t} + \frac{\overline{\partial \bar{u}_i u_i'}}{2\partial t} + \frac{\overline{\partial u_i' u_i'}}{2\partial t} + \bar{u}_i \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \overline{u_i' \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j}} + \overline{\bar{u}_i \frac{\partial u_i' \bar{u}_j}{\partial x_j}} + \overline{u_i' \frac{\partial u_i' \bar{u}_j}{\partial x_j}} + \overline{\bar{u}_i \frac{\partial \bar{u}_i u_j'}{\partial x_j}} \\
&+ \overline{u_i' \frac{\partial \bar{u}_i u_j'}{\partial x_j}} + \overline{\bar{u}_i \frac{\partial u_i' u_j'}{\partial x_j}} + \overline{u_i' \frac{\partial u_i' u_j'}{\partial x_j}} = -\frac{1}{\rho}\left(\bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \overline{u_i' \frac{\partial \bar{p}}{\partial x_i}} + \overline{\bar{u}_i \frac{\partial p'}{\partial x_i}} + \overline{u_i' \frac{\partial p'}{\partial x_i}} \right) \\
&+ \nu \left(\bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \overline{u_i' \frac{\partial^2 \bar{u}_i}{\partial x_j^2}} + \overline{\bar{u}_i \frac{\partial^2 u_i'}{\partial x_j^2}} + \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j^2}} \right)
\end{aligned} \tag{7}$$

Simplify

$$
\begin{aligned}
&\frac{\partial \bar{u}_i \bar{u}_i}{2\partial t} + \overline{\frac{\partial u'_i u'_i}{2\partial t}} + \bar{u}_i \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \overline{u'_i \frac{\partial u'_i \bar{u}_j}{\partial x_j}} + \overline{u'_i \frac{\partial \bar{u}_i u'_j}{\partial x_j}} + \overline{\bar{u}_i \frac{\partial u'_i u'_j}{\partial x_j}} \overline{+ u'_j \frac{\partial u'_j u'_j}{\partial x_j}} = \\
&-\frac{1}{\rho}\left(\bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \overline{u'_i \frac{\partial p'}{\partial x_i}} \right) + \nu \left(\bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial x_j^2} + \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j^2}} \right)
\end{aligned}
\tag{8}
$$

To get a transport equation for the turbulent kinetic energy (the $\overline{u'_i u'_i}$ term), get rid of the mean KE term. Multiply mean momentum equation by $\bar{u}_i$

$$
\begin{aligned}
&\bar{u}_i \left(\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \overline{\frac{\partial u'_i u'_j}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2} \right) = \\
&\frac{\partial \bar{u}_i \bar{u}_i}{2\partial t} + \bar{u}_i \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_j} + \overline{\bar{u}_i \frac{\partial u'_i u'_j}{\partial x_j}} = -\frac{1}{\rho}\bar{u}_i \frac{\partial \bar{p}}{\partial x_i} + \nu \bar{u}_i \frac{\partial^2 \bar{u}_i}{\partial x_j^2}
\end{aligned}
\tag{9}
$$

Subtract this equation from (8)

$$
\overline{\frac{\partial u'_i u'_i}{2\partial t}} + \overline{u'_i \frac{\partial u'_i \bar{u}_j}{\partial x_j}} + \overline{u'_i \frac{\partial \bar{u}_i u'_j}{\partial x_j}} \overline{+ u'_i \frac{\partial u'_i u'_j}{\partial x_j}} = -\frac{1}{\rho}\left(\overline{u'_i \frac{\partial p'}{\partial x_i}} \right) + \nu \left(\overline{u'_i \frac{\partial^2 u'_i}{\partial x_j^2}} \right)
\tag{10}
$$

Rewrite using continuity and $k = \dfrac{u'_i u'_i}{2}$

$$
\overline{\frac{\partial k}{\partial t}} + \bar{u}_j \overline{\frac{\partial k}{\partial x_j}} + \overline{u'_i u'_j \frac{\partial \bar{u}_i}{\partial x_j}} + \overline{\frac{\partial u'_j k}{\partial x_j}} = -\frac{1}{\rho}\left(\overline{u'_i \frac{\partial p'}{\partial x_i}} \right) + \nu \overline{\frac{\partial^2 k}{\partial x_j^2}} - \nu \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}
\tag{11}
$$

which is one form of the Turbulent Kinetic Energy equation. A common rearrangement is

$$
\overline{\frac{\partial k}{\partial t}} + \bar{u}_j \overline{\frac{\partial k}{\partial x_j}} + \overline{u'_i u'_j \frac{\partial \bar{u}_i}{\partial x_j}} + \frac{\partial}{\partial x_j}\left(\overline{u'_j k} + \overline{\frac{u'_j}{\rho} p'} \right) = \nu \left(\overline{\frac{\partial^2 k}{\partial x_j^2}} \right) - \nu \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}}
\tag{12}
$$

From left to right, these terms are interpreted as:

1. Time rate of change of TKE
2. Convection of TKE by mean flow
3. TKE production by Reynolds stress acting on mean velocity gradient
4. Transport of TKE by fluctuating velocity and by pressure-velocity correlation
5. Diffusion of TKE
6. Dissipation of TKE