Met 205A
Assignment 7
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Q3) *Derive an expression for the phase speed of a sound wave, expanding on Holton's procedure in section 7.3*

Sound waves are longitudinal waves which are characterized by the fact that the particle motion they cause is always parallel to the direction of propagation. This class of waves is distinct from transverse waves which are classified by the fact that particle motion is orthogonal to propagation direction. Therefore, in deriving expressions for longitudinal waves, the simplification that there is only motion in the x-direction may be made, which implies that the v and w components of the momentum equations equal 0. Friction and the Coriolis force can also be neglected, thereby producing the following simplified momentum equation:

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{1}$$

The mass continuity can be simplified to:

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \tag{2}$$

Assuming that the flow is adiabatic and that $Q = 0$, the thermodynamic energy equation simplifies to:

$$\frac{D\ln\theta}{Dt} = 0 \tag{3}$$

resulting in three equations with three unknowns $u, p,$ and ρ. Since θ, potential temperature, can be expressed as a function of p and ρ, it can be eliminated in (3) by substituting the terms which equal T in the ideal gas law,

$$T = \frac{p}{R\rho} \tag{4}$$

into Poisson's equation,

$$\theta = T\left(\frac{p_s}{p}\right)^{R/c_p} \tag{5}$$

thereby producing the following equation for potential temperature:

$$\theta = \left(\frac{p}{\rho R}\right)\left(\frac{p_s}{p}\right)^{R/c_p} \tag{6}$$

After substituting the right hand side (RHS) of (6) into (3) and differentiating, θ is eliminated from (3), producing a new form of the thermodynamic energy equation (8) which contains the two dependent variables, p and ρ.

$$\frac{D}{Dt}\left\{\ln\left[\left(\frac{p}{\rho R}\right)\left(\frac{p_s}{p}\right)^{R/c_p}\right]\right\} = 0$$

$$\frac{D}{Dt}\left(\ln\left(\frac{p}{\rho R}\right) + \ln\left(\frac{p_s}{p}\right)^{R/c_p}\right) = 0$$

$$\frac{D}{Dt}\left(\ln\left(\frac{p}{\rho R}\right)\right) + \frac{R}{c_p}\frac{D}{Dt}\ln\left(\frac{p_s}{p}\right) = 0$$

$$\frac{D}{Dt}\ln p - \frac{D}{Dt}\ln(\rho R) + \frac{R}{c_p}\frac{D}{Dt}\ln\left(\frac{p_s}{p}\right) = 0$$

$$\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} + \frac{R}{c_p}\left[\cancel{\frac{D\ln p_s}{Dt}} - \frac{D\ln p}{Dt}\right] = 0 \quad \Big| \; p_s = 1000\text{ hPa} \;\&\; \therefore \; \frac{D\ln p_s}{Dt} = 0$$

$$\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} - \frac{R}{c_p}\frac{D\ln p}{Dt} = 0$$

$$\frac{D\ln p}{Dt}\left(1 - \frac{R}{c_p}\right) - \frac{D\ln\rho}{Dt} = 0 \tag{7}$$

Substituting $R = c_p - c_v$ into (7) produces:

$$\frac{D\ln p}{Dt}\left(1-\frac{c_p-c_v}{c_p}\right)-\frac{D\ln\rho}{Dt}=0$$

$$\frac{D\ln p}{Dt}\left(1-\frac{c_p}{c_p}+\frac{c_v}{c_p}\right)-\frac{D\ln\rho}{Dt}=0$$

$$\frac{D\ln p}{Dt}\left(\frac{c_v}{c_p}\right)-\frac{D\ln\rho}{Dt}=0$$

Recognizing that $\gamma = \frac{c_p}{c_v}$ and substituting above, the thermodynamic energy equation now has the form:

$$\frac{1}{\gamma}\frac{D\ln p}{Dt}-\frac{D\ln\rho}{Dt}=0 \quad \Rightarrow \quad \frac{1}{\gamma}\frac{D\ln p}{Dt}=\frac{D\ln\rho}{Dt} \tag{8}$$

ρ can be eliminated in the continuity equation (2) by dividing both sides by its reciprocal, resulting in

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u}{\partial x}=0, \tag{9}$$

and by recognizing from the following differentiation rule $\frac{d\ln a}{dt}=\frac{1}{a}\frac{da}{dt}$

that the density term of (8) is equivalent to the density term in (9) when rewritten in the following form:

$$\frac{1}{\rho}\frac{D\rho}{Dt}=\frac{D\ln\rho}{Dt} \tag{10}$$

Therefore the RHS of (8) can be replaced with $\frac{\partial u}{\partial x}$, producing:

$$\frac{1}{\gamma}\frac{D\ln p}{Dt}+\frac{\partial u}{\partial x}=0 \tag{11}$$

The new forms of the governing equations can now be linearized by decomposing the dependent variables into their basic state and perturbation components:

$$\begin{aligned} u(x,t) &= \bar{u} + u'(x,t) \\ p(x,t) &= \bar{p} + p'(x,t) \\ \rho(x,t) &= \bar{\rho} + \rho'(x,t) \end{aligned} \tag{12}$$

Substituting (12) into (1) and expanding using the simplified definition of the total derivative, $\frac{D\phi}{Dt} \equiv \frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x}$, (where ϕ represents any appropriate quantity), we obtain:

$$\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + \frac{1}{(\bar{\rho}+\rho')}\frac{\partial}{\partial x}(\bar{p}+p') = 0 \tag{13}$$

which is the linearized version of the *u*-momentum equation.

Substituting (12) into (11) and using above differentiation rule for p $\left(\frac{D\ln p}{Dt} = \frac{1}{p}\frac{Dp}{Dt}\right)$ produces:

$$\frac{1}{\gamma}\frac{1}{(\bar{p}+p')}\left[\frac{\partial}{\partial t}(\bar{p}+p') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{p}+p')\right] + \frac{\partial}{\partial x}(\bar{u}+u') = 0$$

After multiplying by reciprocals, the perturbation method produces the following linearized form of the thermodynamic energy equation:

$$\frac{\partial}{\partial t}(\bar{p}+p') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{p}+p') + \gamma(\bar{p}+p')\frac{\partial}{\partial x}(\bar{u}+u') = 0 \tag{14}$$

Provided that $\left|\frac{\rho'}{\bar{\rho}}\right| \ll 1$, the density term $\left(\frac{1}{(\bar{\rho}+\rho')}\right)$ in (13) can be approximated by using binomial expansion. Recognizing that binomial series of the form $(1+x)^k$ always converge when $|x| < 1$ (Stewart, 2003), and using the expansion definition for a binomial series below,

$$\boxed{(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty}\binom{k}{n}x^n} \quad \text{where,} \quad x = \frac{\rho'}{\bar{\rho}},\ k = -1$$

$$\frac{1}{\rho} = \frac{1}{(\bar{\rho}+\rho')} = \frac{1}{\bar{\rho}\left(1+\frac{\rho'}{\bar{\rho}}\right)} = \frac{1}{\bar{\rho}}\left(1+\frac{\rho'}{\bar{\rho}}\right)^{-1} \text{ expands to } \Rightarrow$$

$$\frac{1}{\bar{\rho}}\left(1\left(\frac{\rho'}{\bar{\rho}}\right)^0 + (-1)\frac{\rho'}{\bar{\rho}} + \ldots\right) \Rightarrow \frac{1}{\bar{\rho}}\left(1 - \frac{\rho'}{\bar{\rho}} + \frac{2}{2!}\left(\frac{\rho'}{\bar{\rho}}\right)^2 - \frac{6}{3!}\left(\frac{\rho'}{\bar{\rho}}\right)^2 + \ldots\right) \Rightarrow$$

it can be seen that the expanding terms will increasingly approach zero as the factorials in the denominator increase. Therefore, $\frac{1}{\bar{\rho}}\left(1+\frac{\rho'}{\bar{\rho}}\right)^{-1}$ will be approximately equal $\frac{1}{\bar{\rho}}\left(1-\frac{\rho'}{\bar{\rho}}\right)$.

The terms to the right of $\left(1-\frac{\rho'}{\bar{\rho}}\right)$, i.e. 2! term, 3! term, etc…, will be very small and can be neglected. Thus,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow \frac{1}{\bar{\rho}}\left(1-\frac{\rho'}{\bar{\rho}}\right)\left(\cancel{\frac{\partial \bar{p}}{\partial x}} + \frac{\partial p'}{\partial x}\right) \approx \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}.$$

$\frac{\partial \bar{p}}{\partial x}$ becomes zero upon differentiating since the mean component, $\bar{p}$, is constant and does not vary in the x-direction $\left(\text{i.e. } \frac{d}{dx}(\bar{c}) = 0, \text{ where } \bar{c} \text{ is mean component and } \therefore \text{ constant}\right)$. Thus the linearized form of the u-momentum equation (14) becomes:

$$\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + \frac{1}{\bar{\rho}}\frac{\partial}{\partial x}(\bar{p}+p') = 0$$

$$\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + \cancel{\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x}} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(\bar{u}+u') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u') + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \qquad (15)$$

Holton defines the fundamental assumptions of Linear Perturbation Theory as follows:

- Basic state variables must themselves satisfy the governing equations when the perturbations are set to zero.
- The perturbation fields must be small enough so that all terms in the governing equations that involve products of perturbations can be neglected.

We can now further simplify (16) $\frac{\partial}{\partial t}(\bar{u}+u')+(\bar{u}+u')\frac{\partial}{\partial x}(\bar{u}+u')+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0$

by differentiating and applying the above assumptions to arrive at the u-momentum linear perturbation equation (16):

$$\cancel{\frac{\partial \bar{u}}{\partial t}}+\frac{\partial u'}{\partial t}+\bar{u}\cancel{\frac{\partial \bar{u}}{\partial x}}+\bar{u}\frac{\partial u'}{\partial x}+u'\cancel{\frac{\partial \bar{u}}{\partial x}}+u'\cancel{\frac{\partial u'}{\partial x}}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0$$

$$\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0$$

$$\boxed{\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0} \qquad (16)$$

The above techniques can be applied to linearize (13) and produce the thermodynamic linear perturbation equation (17):

$$\frac{\partial}{\partial t}(\bar{p}+p')+(\bar{u}+u')\frac{\partial}{\partial x}(\bar{p}+p')+\gamma(\bar{p}+p')\frac{\partial}{\partial x}(\bar{u}+u')=0$$
$$\frac{\partial \bar{p}}{\partial t}+\frac{\partial p'}{\partial t}+(\bar{u}+u')\left[\frac{\partial \bar{p}}{\partial x}+\frac{\partial p'}{\partial x}\right]+\gamma(\bar{p}+p')\left[\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}\right]=0$$
$$\frac{\partial \bar{p}}{\partial t}+\frac{\partial p'}{\partial t}+\bar{u}\frac{\partial \bar{p}}{\partial x}+\bar{u}\frac{\partial p'}{\partial x}+u'\frac{\partial \bar{p}}{\partial x}+u'\frac{\partial p'}{\partial x}+(\gamma\bar{p}+\gamma p')\left[\frac{\partial \bar{u}}{\partial x}+\frac{\partial u'}{\partial x}\right]=0$$
$$\cancel{\frac{\partial \bar{p}}{\partial t}}+\frac{\partial p'}{\partial t}+\bar{u}\cancel{\frac{\partial \bar{p}}{\partial x}}+\bar{u}\frac{\partial p'}{\partial x}+u'\cancel{\frac{\partial \bar{p}}{\partial x}}+u'\cancel{\frac{\partial p'}{\partial x}}+\gamma\cancel{\bar{p}\frac{\partial \bar{u}}{\partial x}}+\gamma\bar{p}\frac{\partial u'}{\partial x}+\gamma\cancel{p'\frac{\partial \bar{u}}{\partial x}}+\gamma\cancel{p'\frac{\partial u'}{\partial x}}=0$$

$$\frac{\partial p'}{\partial t}+\bar{u}\frac{\partial p'}{\partial x}+\gamma\bar{p}\frac{\partial u'}{\partial x}=0 \quad \Rightarrow$$
$$\boxed{\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)p'+\gamma\bar{p}\frac{\partial u'}{\partial x}=0} \qquad (17)$$

Holton eliminates u' by operating on (17) with $\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)$ and substituting from (16):

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)p'+\gamma\bar{p}\frac{\partial u'}{\partial x}\right]=\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)0 \Rightarrow$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 p' + \left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\gamma\bar{p}\frac{\partial u'}{\partial x}=0 \Rightarrow$$

Apply chain rule and linearization rules:

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 p' + \cancel{\frac{\partial}{\partial t}(\gamma\bar{p})\frac{\partial u'}{\partial x}} + \gamma\bar{p}\frac{\partial^2 u'}{\partial t\partial x} + \cancel{\bar{u}\frac{\partial}{\partial x}(\gamma\bar{p})\frac{\partial u'}{\partial x}} + \bar{u}\gamma\bar{p}\frac{\partial^2 u'}{\partial x^2}=0$$

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 p' + \gamma\bar{p}\frac{\partial^2 u'}{\partial t\partial x} + \bar{u}\gamma\bar{p}\frac{\partial^2 u'}{\partial x^2}=0 \tag{18}$$

Expand 1st term in (18). Holton notes that the squared differential operator in the first term expands in the usual way, i.e.:

$$\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)^2 \Rightarrow \frac{\partial^2}{\partial t^2}+2\bar{u}\frac{\partial^2}{\partial t\partial x}+\bar{u}^2\frac{\partial^2}{\partial x^2}$$

Rewrite (18) with above expanded term:

$$\left[\frac{\partial^2}{\partial t^2}+2\bar{u}\frac{\partial^2}{\partial t\partial x}+\bar{u}^2\frac{\partial^2}{\partial x^2}\right]p' + \gamma\bar{p}\frac{\partial^2 u'}{\partial t\partial x} + \bar{u}\gamma\bar{p}\frac{\partial^2 u'}{\partial x^2}=0 \tag{19}$$

Eliminate u' by operating on (16) with $\gamma\bar{p}\frac{\partial}{\partial x}$

$$\gamma\bar{p}\frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0\right) \Rightarrow \gamma\bar{p}\frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)u'+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0\right)\Rightarrow$$

$$\gamma\bar{p}\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial t}+\bar{u}\frac{\partial u'}{\partial x}+\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}=0\right)\Rightarrow \gamma\bar{p}\frac{\partial^2 u'}{\partial x\partial t}+\cancel{\gamma\bar{p}\frac{\partial\bar{u}}{\partial x}\frac{\partial u'}{\partial x}}+\gamma\bar{p}\bar{u}\frac{\partial^2 u'}{\partial x^2}+\gamma\bar{p}\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2}=0$$

$$\gamma\bar{p}\frac{\partial^2 u'}{\partial x\partial t}+\gamma\bar{p}\bar{u}\frac{\partial^2 u'}{\partial x^2}+\gamma\bar{p}\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2}=0 \tag{20}$$

Add (19) and (20) to eliminate u'

$$\frac{\partial^2 p'}{\partial t^2} + 2\bar{u}\frac{\partial^2 p'}{\partial t \partial x} + \bar{u}^2 \frac{\partial^2 p'}{\partial x^2} + \cancel{\gamma \bar{p}\frac{\partial^2 u'}{\partial t \partial x}} + \cancel{\bar{u}\gamma \bar{p}\frac{\partial^2 u'}{\partial x^2}} = 0$$

$$- \quad \cancel{\gamma \bar{p}\frac{\partial^2 u'}{\partial x \partial t}} + \qquad \cancel{\gamma \bar{p}\bar{u}\frac{\partial^2 u'}{\partial x^2}} + \gamma \bar{p}\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0$$

$$\Rightarrow \quad \frac{\partial^2 p'}{\partial t^2} + 2\bar{u}\frac{\partial^2 p'}{\partial t \partial x} + \bar{u}^2 \frac{\partial^2 p'}{\partial x^2} - \gamma \bar{p}\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0$$

$$\frac{\partial^2 p'}{\partial t^2} + 2\bar{u}\frac{\partial^2 p'}{\partial t \partial x} + \bar{u}^2 \frac{\partial^2 p'}{\partial x^2} - \gamma \bar{p}\frac{1}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0 \quad \Rightarrow$$

Rewrite 1st three terms on LHS in compressed form {as in (19)}

$$\boxed{\left(\frac{\partial^2}{\partial t^2} + \bar{u}\frac{\partial}{\partial x}\right)^2 p' - \frac{\gamma \bar{p}}{\bar{\rho}}\frac{\partial^2 p'}{\partial x^2} = 0} \tag{21}$$

(21) is a general form of the standard wave equation.

The three simplified governing equations (1), (2), and (3) which had three unknowns, u, p, and ρ, were linearized with the perturbation method and reduced down to one equation (21) with one unknown, p'. Therefore solving for p' in (21) alows one to subsequently solve for u' and ρ'.

Since (21) is a general form of the standard wave equation, one may assume/test that there is solution in the general form: $p' = Ae^{ik(x-ct)}$, where only its real component,

$$p' = \mathrm{Re}\left\{Ae^{ik(x-ct)}\right\} \tag{22}$$

would have physical meaning. (22) represents a sinusoidal wave.

Test the solution by substituting it into (21):

$$\left(\frac{\partial^2}{\partial t^2} + \bar{u}\frac{\partial}{\partial x}\right)^2 Ae^{ik(x-ct)} - \frac{\gamma \bar{p}}{\bar{\rho}}\frac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} = 0$$

$$\frac{\partial^2 Ae^{ik(x-ct)}}{\partial t^2} + 2\bar{u}\frac{\partial^2 Ae^{ik(x-ct)}}{\partial t \partial x} + \bar{u}^2\frac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} - \frac{\gamma \bar{p}}{\bar{\rho}}\frac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} = 0 \tag{23}$$

Differentiate each term from LHS of (23):

Term 1: $\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial t^2} \Rightarrow \dfrac{\partial Ae^{ik(x-ct)}}{\partial t} \Rightarrow A\dfrac{\partial e^{ikx-ikct}}{\partial t} \Rightarrow -Aikce^{ik(x-ct)} \Rightarrow$

$\dfrac{\partial}{\partial t}\left(-Aikce^{ikx-ikct}\right) \Rightarrow -Aikc\cdot(-ikc)e^{ik(x-ct)} \Rightarrow \boxed{\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial t^2} = -Ak^2c^2e^{ik(x-ct)}}$

Term 2: $2\bar{u}\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial t\partial x} \Rightarrow 2\bar{u}\dfrac{\partial}{\partial x}\left(-Aikce^{ik(x-ct)}\right) \Rightarrow -2\bar{u}Aikcike^{ik(x-ct)} \Rightarrow$ | Use $\dfrac{\partial}{\partial t}$ result from above

$\Rightarrow \boxed{2\bar{u}\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial t\partial x} = 2\bar{u}Ak^2ce^{ik(x-ct)}}$

Term 3: $\bar{u}^2\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} \Rightarrow \bar{u}^2\dfrac{\partial}{\partial x}\left(Aike^{ik(x-ct)}\right) \Rightarrow \bar{u}^2 Aikike^{ik(x-ct)} \Rightarrow$

$\Rightarrow \boxed{\bar{u}^2\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} = -\bar{u}^2Ak^2e^{ik(x-ct)}}$

Term 4: $-\dfrac{\gamma\bar{p}}{\bar{\rho}}\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} \Rightarrow -\dfrac{\gamma\bar{p}}{\bar{\rho}}\left(-Ak^2e^{ik(x-ct)}\right) \Rightarrow \boxed{-\dfrac{\gamma\bar{p}}{\bar{\rho}}\dfrac{\partial^2 Ae^{ik(x-ct)}}{\partial x^2} = \dfrac{\gamma\bar{p}}{\bar{\rho}}Ak^2e^{ik(x-ct)}}$

Therefore, recombining the four tems above produces:

$$-Ak^2c^2e^{ik(x-ct)} + 2\bar{u}Ak^2ce^{ik(x-ct)} + \left(-\bar{u}^2Ak^2e^{ik(x-ct)}\right) - \frac{\gamma\bar{p}}{\bar{\rho}}Ak^2e^{ik(x-ct)} = 0$$

Divide through by $Ae^{ik(x-ct)}$ and solve for c

$$-k^2c^2 + 2\bar{u}k^2c - \bar{u}^2k^2 + \frac{\gamma\bar{p}}{\bar{\rho}}k^2 = 0 \quad\Rightarrow\quad k^2\left(-c^2 + 2\bar{u}c - \bar{u}^2 + \frac{\gamma\bar{p}}{\bar{\rho}}\right) = 0 \quad\Rightarrow$$

$$-c^2 + 2\bar{u}c - \bar{u}^2 + \frac{\gamma\bar{p}}{\bar{\rho}} = 0 \quad\Rightarrow\quad c^2 - 2\bar{u}c + \bar{u}^2 = \frac{\gamma\bar{p}}{\bar{\rho}} \quad\Rightarrow$$

$$(c-\bar{u})^2 = \frac{\gamma\bar{p}}{\bar{\rho}} \quad\Rightarrow$$

$$\boxed{c = \bar{u} \pm \sqrt{\frac{\gamma\bar{p}}{\bar{\rho}}}} \qquad (24)$$

Using the ideal gas law (4), (24) can be written in the form

$$\boxed{c = \bar{u} \pm \sqrt{\gamma R\bar{T}}} \tag{25}$$

Holton states that (24) is a solution to (21) if the phase speed, *c*, satisfies (24). This can be tested using values for standard conditions.

$c = \bar{u} \pm \sqrt{\gamma R\bar{T}}$

$\bar{u} = 10 \text{ m s}^{-1}$

$\gamma = \dfrac{c_p}{c_v} = 1.4$ for dry air

$R = R_d = 287.05 \text{ J kg}^{-1} \text{ K}^{-1}$

$\bar{T} = 293\text{K}$

$$c = \bar{u} \pm \sqrt{\gamma R\bar{T}} \Rightarrow 10 \text{ m s}^{-1} \pm \sqrt{(1.4)\left(287.05 \text{ J kg}^{-1} \text{ K}^{-1}\right)(293\text{K})},$$

$$\left[\sqrt{\text{N m kg}^{-1} \text{ K}^{-1} \text{ K}}\right] \Rightarrow \left[\sqrt{\cancel{\text{kg}} \text{ m}^2 \text{ s}^{-2} \cancel{\text{kg}^{-1}} \cancel{\text{K}^{-1}} \cancel{\text{K}}}\right] \Rightarrow \left[\text{m s}^{-1}\right]$$

$c = 10 \text{ m s}^{-1} \pm 343 \text{ m s}^{-1}$

Thus, the units are correct and the value arrived at is precisely the speed of sound when compared with various reference values.