Reynolds Stress Transport Equation

Start with Navier Stokes Equations (incompressible flow)

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1}$$

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial u_l}{\partial x_l} \right) \tag{2}$$

and mean momentum equations (incompressible flow)

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \qquad \frac{\partial u'_i}{\partial x_i} = 0 \tag{3}$$

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2} - \frac{\partial (\overline{u'_i u'_j})}{\partial x_j} \tag{4}$$

Use instantaneous Navier-Stokes Equations (incompressible flow)

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j} \right) \tag{5}$$

multiply by u_j

$$u_j \left(\rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k} \right) \right) \tag{6}$$

multiply by u_i

$$u_i \left(\rho \frac{\partial u_j}{\partial t} + \rho u_l \frac{\partial u_j}{\partial x_l} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_l}\left(\mu \frac{\partial u_j}{\partial x_l} \right) \right) \tag{7}$$

Add eqns (6) and (7)

$$u_i \rho \frac{\partial u_j}{\partial t} + \rho u_i u_l \frac{\partial u_j}{\partial x_l} + u_j \rho \frac{\partial u_i}{\partial t} + \rho u_j u_k \frac{\partial u_i}{\partial x_k} = -u_i \frac{\partial p}{\partial x_j} + u_i \frac{\partial}{\partial x_l}\left(\mu \frac{\partial u_j}{\partial x_l} \right) - u_j \frac{\partial p}{\partial x_i} + u_j \frac{\partial}{\partial x_k}\left(\mu \frac{\partial u_i}{\partial x_k} \right) \tag{8}$$

or

$$\frac{\partial u_i u_j}{\partial t} + u_i u_l \frac{\partial u_j}{\partial x_l} + u_j u_k \frac{\partial u_i}{\partial x_k} = -\frac{u_i}{\rho}\frac{\partial p}{\partial x_j} + u_i \nu \frac{\partial^2 u_j}{\partial x_l^{\,2}} - \frac{u_j}{\rho}\frac{\partial p}{\partial x_i} + u_j \frac{\partial^2 u_i}{\partial x_k^{\,2}} \tag{9}$$

Apply Reynolds decomposition

$$u_i = \overline{u}_i + u'_i \tag{10}$$

$$\frac{\partial(\bar{u}_i+u'_i)(\bar{u}_j+u'_j)}{\partial t}+(\bar{u}_i+u'_i)(\bar{u}_l+u'_l)\frac{\partial(\bar{u}_j+u'_j)}{\partial x_l}+(\bar{u}_j+u'_j)(\bar{u}_k+u'_k)\frac{\partial(\bar{u}_i+u'_i)}{\partial x_k}=$$
$$-\frac{(\bar{u}_i+u'_i)}{\rho}\frac{\partial(\bar{p}+p')}{\partial x_j}+(\bar{u}_i+u'_i)\nu\frac{\partial^2(\bar{u}_j+u'_j)}{\partial x_l^{\,2}}-\frac{(\bar{u}_j+u'_j)}{\rho}\frac{\partial(\bar{p}+p')}{\partial x_i}+(\bar{u}_j+u'_j)\nu\frac{\partial^2(\bar{u}_i+u'_i)}{\partial x_k^2} \tag{11}$$

Simplify term by term and average

$$\overline{\frac{\partial(\bar{u}_i+u'_i)(\bar{u}_j+u'_j)}{\partial t}}=\overline{\frac{\partial}{\partial t}\left(\overline{u_i u_j}+\bar{u}_i u'_j+\bar{u}_j u'_i+u'_i u'_j\right)}=\frac{\partial}{\partial t}\left(\overline{u_i u_j}+\overline{u'_i u'_j}\right)$$

$$\overline{(\bar{u}_i+u'_i)(\bar{u}_l+u'_l)\frac{\partial(\bar{u}_j+u'_j)}{\partial x_l}}=\overline{\left(\overline{u_i u_l}+\bar{u}_i u'_l+\bar{u}_l u'_i+u'_i u'_l\right)\frac{\partial(\bar{u}_j+u'_j)}{\partial x_l}}=$$
$$\left(\overline{u_i u_l}+\overline{u'_i u'_l}\right)\frac{\partial\bar{u}_j}{\partial x_l}+\overline{(\bar{u}_i u'_l+\bar{u}_l u'_i+u'_i u'_l)\frac{\partial u'_j}{\partial x_l}}$$

$$\overline{(\bar{u}_j+u'_j)(\bar{u}_k+u'_k)\frac{\partial(\bar{u}_i+u'_i)}{\partial x_k}}=\overline{(\bar{u}_j\bar{u}_k+\bar{u}_j u'_k+\bar{u}_k u'_j+u'_k u'_j)}\overline{\frac{\partial(\bar{u}_i+u'_i)}{\partial x_k}}=$$
$$\left(\bar{u}_j\bar{u}_k+\overline{u'_k u'_j}\right)\frac{\partial\bar{u}_i}{\partial x_k}+\overline{\bar{u}_j u'_k+\bar{u}_k u'_j+u'_k u'_j\frac{\partial u'_i}{\partial x_k}}$$

$$\overline{\frac{(\bar{u}_i+u'_i)}{\rho}\frac{\partial(\bar{p}+p')}{\partial x_j}}=\frac{1}{\rho}\left(\bar{u}_i\frac{\partial\bar{p}}{\partial x_j}+\overline{u'_i\frac{\partial p'}{\partial x_j}}\right)$$

$$\overline{(\bar{u}_i+u'_i)\nu\frac{\partial^2(\bar{u}_j+u'_j)}{\partial x_l^{\,2}}}=\nu\bar{u}_i\frac{\partial^2\bar{u}_j}{\partial x_l^{\,2}}+\nu\overline{u'_i\frac{\partial^2 u'_j}{\partial x_l^{\,2}}}$$

$$\overline{\frac{(\bar{u}_j+u'_j)}{\rho}\frac{\partial(\bar{p}+p')}{\partial x_i}}=\frac{1}{\rho}\left(\bar{u}_j\frac{\partial\bar{p}}{\partial x_i}+\overline{u'_j\frac{\partial p'}{\partial x_i}}\right)$$

$$\overline{\nu(\bar{u}_j+u'_j)\frac{\partial^2(\bar{u}_i+u'_i)}{\partial x_k^2}}=\nu\bar{u}_j\frac{\partial^2\bar{u}_i}{\partial x_k^2}+\nu\overline{u'_j\frac{\partial^2 u'_i}{\partial x_k^2}}$$

Combine

$$\overline{\frac{\partial}{\partial t}\left(\overline{u_i u_j}+\overline{u'_i u'_j}\right)}+\left(\overline{u_i u_l}+\overline{u'_i u'_l}\right)\frac{\partial\bar{u}_j}{\partial x_l}$$
$$+\overline{\left(\bar{u}_i u'_l+\bar{u}_l u'_i+u'_i u'_l\right)\frac{\partial u'_j}{\partial x_l}}+\left(\bar{u}_j\bar{u}_k+\overline{u'_k u'_j}\right)\frac{\partial\bar{u}_i}{\partial x_k}+\overline{\left(\bar{u}_j u'_k+\bar{u}_k u'_j+u'_k u'_j\right)\frac{\partial u'_i}{\partial x_k}}- \tag{12}$$
$$\frac{1}{\rho}\left(\bar{u}_i\frac{\partial\bar{p}}{\partial x_j}+\overline{u'_i\frac{\partial p'}{\partial x_j}}\right)-\frac{1}{\rho}\left(\bar{u}_j\frac{\partial\bar{p}}{\partial x_i}+\overline{u'_j\frac{\partial p'}{\partial x_i}}\right)+\nu\bar{u}_i\frac{\partial^2\bar{u}_j}{\partial x_l^2}+\nu\overline{u'_i\frac{\partial^2 u'_j}{\partial x_l^2}}+\nu\bar{u}_j\frac{\partial^2\bar{u}_i}{\partial x_k^2}+\nu\overline{u'_j\frac{\partial^2 u'_i}{\partial x_k^2}}$$

Make a similar equation from the mean momentum equation, by multiplying $\frac{\partial \bar{u}_i}{\partial t} + \ldots$ by $\bar{u}_j$ and multiplying $\bar{u}_i$ by $\bar{u}_i$ and adding.

$$\bar{u}_j \left(\rho \frac{\partial \bar{u}_i}{\partial t} + \rho u_k \frac{\partial \bar{u}_i}{\partial x_k} = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\mu \frac{\partial \bar{u}_i}{\partial x_k} - \overline{u'_i u'_j} \delta_{jk} \right) \right) \tag{13}$$

$$\bar{u}_i \left(\rho \frac{\partial \bar{u}_j}{\partial t} + \rho u_l \frac{\partial \bar{u}_j}{\partial x_l} = -\frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_l}\left(\mu \frac{\partial \bar{u}_j}{\partial x_l} - \overline{u'_i u'_j} \delta_{il} \right) \right) \tag{14}$$

$$\begin{aligned} &\frac{\partial \overline{u_i u_j}}{\partial t} + \overline{u_k u_j} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_l u_i} \frac{\partial \bar{u}_j}{\partial x_l} = -\frac{\bar{u}_j}{\rho} \frac{\partial \bar{p}}{\partial x_i} - \frac{\bar{u}_i}{\rho} \frac{\partial \bar{p}}{\partial x_j} + \\ &\bar{u}_j \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \bar{u}_i}{\partial x_k} - \overline{u'_i u'_k} \right) + \bar{u}_i \frac{\partial}{\partial x_l}\left(\nu \frac{\partial \bar{u}_j}{\partial x_l} - \overline{u'_l u'_j} \right) \end{aligned} \tag{15}$$

Subtract eqn (15) from eqn (12)

$$\begin{aligned} &\frac{\partial}{\partial t}(\overline{u'_i u'_j}) + \overline{u'_i u'_l} \frac{\partial \bar{u}_j}{\partial x_l} + \overline{(\bar{u}_i u'_l + \bar{u}_l u'_i + u'_i u'_l) \frac{\partial u'_j}{\partial x_l}} + \overline{u'_k u'_j} \frac{\partial \bar{u}_i}{\partial x_k} + \overline{\bar{u}_j u'_k + \bar{u}_k u'_j + u'_k u'_j \frac{\partial u'_i}{\partial x_k}} = \\ &-\frac{1}{\rho} \overline{u'_i \frac{\partial p'}{\partial x_j}} - \frac{1}{\rho} \overline{u'_j \frac{\partial p'}{\partial x_i}} + \nu \overline{u'_i \frac{\partial^2 u'_j}{\partial x_l^2}} + \nu \overline{u'_j \frac{\partial^2 u'_i}{\partial x_k^2}} + \bar{u}_j \frac{\partial \overline{u'_i u'_k}}{\partial x_k} + \bar{u}_i \frac{\partial \overline{u'_l u'_j}}{\partial x_l} \end{aligned}$$

Simplify

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\overline{u'_i u'_j}\right) + \bar{u}_l \frac{\partial \overline{u'_i u'_j}}{\partial x_l} = -\overline{u'_i u'_l} \frac{\partial \bar{u}_j}{\partial x_l} - \overline{u'_k u'_j} \frac{\partial \bar{u}_i}{\partial x_k} - 2\nu \overline{\frac{\partial u'_i}{\partial x_k} \frac{\partial u'_j}{\partial x_k}} + \overline{\frac{p'}{\rho}\left(\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j} \right)} \\ &-\frac{\partial}{\partial x_k}\left(u'_j u'_i u'_k + \overline{u'_i \frac{p'}{\rho}} \delta_{jk} + \overline{u'_j \frac{p'}{\rho}} \delta_{ik} - \nu \frac{\partial \overline{u'_i u'_j}}{\partial x_k} \right) \end{aligned} \tag{16}$$