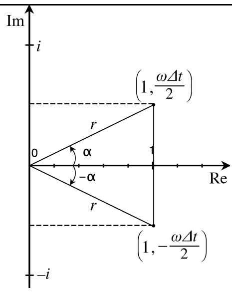
NUMERICAL MODELING REVIEW



COMPLEX NUMBERS

$z = x + iy$

$x = r\cos(\theta);\ \ y = r\sin(\theta)$

$r = \sqrt{x^2 + y^2};\ \ \theta = \arctan\left(\frac{y}{x}\right)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\boxed{x + iy = re^{i\theta}}$, $\boxed{r = \sqrt{x^2 + y^2}}$, $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right) = \sin^{-1}\left(\frac{y}{r}\right)}$

$(x+iy)^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta} = r^n\left(\cos n\theta + i\sin n\theta\right)$

I) LINEAR DIFFERENTIAL EQUATIONS

A DE is an equation of the form: $L(y) = f$, where L is a differential operator and f is a function of the independent variables. The operator, L, is said to be linear if $L(\alpha y_1 + \beta y_2) = \alpha L(y_1) + \beta L(y_2)$ for any 2 functions y_1 & y_2 and 2 scalars α & β.

- ODE $\Rightarrow$ there is 1 independent variable
- PDE $\Rightarrow$ thhere are 2 or more.
- Homogenous $\Rightarrow$ If the operator is linear and the function f is identically 0

Theorem 1: A linear combination of solutions is also a solution
Theorem 2: ...
$\therefore$ Solutions of linear DEs can be constructed by summation

A) LINEAR ODEs

$$\left.\begin{array}{l}\frac{dy}{dt} = y \\ y(0) = y_0\end{array}\right\} \left| \begin{array}{l}\frac{dy}{y} = dt \rightarrow \int\frac{dy}{y} = \int dt \rightarrow \ln y = t + c \rightarrow e^{\ln y} = e^{t+c} \rightarrow \\ \rightarrow \boxed{y(t) = y_0 e^t}\end{array}\right.$$

$$\left.\begin{array}{l}\frac{dy}{dt} = -ky \\ y(0) = y_0\end{array}\right\} \left\{ \frac{dy}{y} = -kdt \rightarrow \int\frac{dy}{y} = -k\int dt \rightarrow \boxed{y(t) = y_0 e^{-kt}}\right.$$

B) LINEAR PDEs

1. Two variables

a. Linearized advection equation (AKA one-way eqn)

$\frac{\partial Q}{\partial t} + c\frac{\partial Q}{\partial x} = 0 \Rightarrow$ | $Q(x,t) = f(x - ct)$ is the general solution

To get unique soluton, IC must be specified, such as: $Q(x,0) = f(x)$

Note: solution is in the form of a traveling disturbance, initial disturbance travels to the right if $c > 0$ or to the left if $c < 0$ w/out change of shape.

II) LINEAR DIFFERENCE EQUATIONS

Difference equations are essentially discrete analogues to differential eqns. In the case of difference eqns, the unknown is not a function, but a sequence. (Actually, a sequence is a function whose domain consists of integers).

- Arithmetic Seq.: $y_{n+1} - y_n = a,\ n = 0,1,2,\ldots$ | Divergent except when $a = 0$
- Geometric Seq.: $y_{n+1} = \lambda y_n,\ n = 0,1,2,\ldots \Rightarrow$ Solution is $\boxed{y_n = \lambda^n y_0}$
  - $\rightarrow |\lambda| < 1 \Rightarrow$ sequnce converges to 0
  - $\rightarrow |\lambda| > 1 \Rightarrow$ sequence diverges

General k^{th} order linear, homogenous difference equation w/ const. coeff:

$\boxed{a_k y_{n+k} + a_{k-1}y_{n+k-1} + \ldots + a_0 y_n = 0,\ n = 0,1,2,\ldots} \rightarrow$ Soln of form: $\boxed{y_n = a\lambda^n}$

Solutions may be found assuming that $y_n = a\lambda^n$, and then solving the roots of the resulting polynomial eqn. If there are k distinct roots, $\lambda_1, \ldots, \lambda_k$, then $\boxed{y_n = c_1\lambda_1^n + c_k\lambda_k^n}$ is the general solution

NUMERICAL METHODS

method is stable/unstable; solution is bounded/unbounded

Review: Waves

$P\left[\text{s cycle}^{-1}\right]$ = period = time for 1 cycle. $\boxed{P = \frac{1}{f}}$

$f\left[\text{Hz} = \text{cycles s}^{-1}\right]$ = # of cycles in a unit time. (Inverse time) | $\boxed{f = \frac{1}{P}}$

$\omega\left[\text{rad s}^{-1}\right]$ = angular frequency. $\boxed{\omega = 2\pi f} \Leftrightarrow \boxed{\omega = \frac{2\pi}{P}}$

Oscillation Equation: $\frac{\partial y}{\partial t} = i\omega y,\ y(0)=y_0 \Rightarrow \boxed{y = y_0 e^{iwy}}$

$y = y_0(\cos\omega t + i\sin\omega t)$ Period: $\frac{2\pi}{\omega}$;

2 level schemes: seek solutions of the form $\boxed{y_n = \lambda^n y_0}$

1) Euler's $\boxed{y_n = (1+i\omega\Delta t)^n y_0}$ $\boxed{\lambda = 1+i\omega\Delta t}$

$\frac{y_{n+1}-y_n}{\Delta t} = i\omega y_n \Rightarrow y_{n+1} = (1+i\omega\Delta t)y_n \Rightarrow \boxed{y_n = (1+i\omega\Delta t)^n y_0}$

We want the absolute value to see if it blows up: $|y_n| = |(1+i\omega\Delta t)^n||y_0|$

Note: $\boxed{|x+iy| = \sqrt{x^2+y^2}} \Rightarrow \boxed{|1+i\omega\Delta t| = \sqrt{1+\omega^2\Delta t^2}}$ | $\omega^2\Delta t^2$ is always >0

- unconditionally unstable

2) Backward $\boxed{y_n = \left(\frac{1}{1-i\omega\Delta t}\right)y_0}$ $\boxed{\lambda = \frac{1}{1-i\omega\Delta t}}$

$\frac{y_{n+1}-y_n}{\Delta t} = f(y_{n+1}) \rightarrow y_{n+1} = \left(\frac{1}{1-i\omega\Delta t}\right)y_n \rightarrow \boxed{y_n = \left(\frac{1}{1-i\omega\Delta t}\right)y_0}$

$|\lambda| = \frac{1}{|1-i\omega\Delta t|} = \frac{1}{\sqrt{1-i\omega\Delta t}}$ & $\therefore$ always <1 decays exponentially

3) Trapezoidal Implicit

$\frac{y_{n+1}-y_n}{\Delta t} = i\omega\left(\frac{y_n+y_{n+1}}{2}\right) \Rightarrow y_n = \lambda^n y_0,\ \lambda = \left(\frac{1+\frac{i\omega\Delta t}{2}}{1-\frac{i\omega\Delta t}{2}}\right)$

$\lambda = e^{2i\alpha},\ \alpha = \arctan\left(\frac{\omega\Delta t}{2}\right)$

LF $\boxed{y_{n+1} = y_{n-1} + 2\Delta t f(y_n)} \Rightarrow \boxed{y_{n+1} = y_{n-1} + 2i\omega\Delta t y_n}$

- seek solutions of the form $y_n = r^n$
- subst: $r^{n+1} = r^{n-1} + (2i\omega\Delta t)(r^n)$, divide by lowest power: r^{n-1}

$r_\pm = i\omega\Delta t \pm \sqrt{1-\omega^2\Delta t^2} \Rightarrow$ Gen Soln: $\boxed{y_n = ar_+^n + br_-^n}$

Case 1: $\omega\Delta t > 1$

$r_\pm = i\omega\Delta t \pm \sqrt{(-1)(\omega^2\Delta t^2 - 1)} \Rightarrow r_\pm = i\omega\Delta t \pm i\sqrt{(\omega^2\Delta t^2-1)} \Rightarrow$

$r_\pm = i\left(\omega\Delta t \pm \sqrt{(\omega^2\Delta t^2-1)}\right)$ |Both #s are on the Img axis.

Since $|i|=1$, take Abs value $|r_\pm| = \left|\omega\Delta t \pm \sqrt{(\omega^2\Delta t^2-1)}\right|$.

If $\omega\Delta t > 1$, Leapfrog (apl to osc eqn) is numerically unstable

$\therefore$ for given ω, Δt can not be too big. $\therefore$ Try smalller values of Δt

Case 2: $\omega\Delta t < 1$, $r_\pm = i\omega\Delta t \pm \sqrt{1-\omega^2\Delta t^2}$ (under rad is Real b/c >0)

- Try squaring both sides: $|r_\pm|^2 = \omega^2\Delta t^2 + (1-\omega^2\Delta t^2) \Rightarrow \boxed{|r_\pm|^2 = 1}$

This happpens because it is bounded: $y_n = ar_+^n + br_-^n \Rightarrow$

ie @$r=1$ $|y_n| \le |a|+|b|$; $|a|+|b|$ will never be bigger than

If $\omega\Delta t < 1$, Leapfrog (apl to osc eqn) is numerically stable

$a = \frac{1+\sqrt{1-\omega^2\Delta t^2}}{2\sqrt{1-\omega^2\Delta t^2}}(y_0),\quad b = \frac{-1+\sqrt{1-\omega^2\Delta t^2}}{2\sqrt{1-\omega^2\Delta t^2}}(y_0)$

Advection

$\left.\begin{array}{l} u(x,0) = Ae^{ikx} \\ u(x,t) = Ae^{ik(x-ct)} \end{array}\right\}(1)$ $\boxed{\frac{\partial u}{\partial x} \cong \frac{u(x+\Delta x,t)-u(x-\Delta x,t)}{2\Delta x} \equiv \delta_{2x}u}$

Exact Soln: $u(x,t) = A_0 e^{ik(x-ct)}$ rewrite as: $u(x,t) = (A_0 e^{-ikct})e^{ikx}$

$\delta_{2x}(e^{ikx}) \Rightarrow \left(\frac{e^{ikx+\Delta x} - e^{-ikx+\Delta x}}{2\Delta x}\right)e^{ikx} = \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right)e^{ikx} \Rightarrow$

$\boxed{\delta_{2x}(e^{ikx}) = i\left(\frac{\sin k\Delta x}{\Delta x}\right)e^{ikx}}$ $\boxed{\text{Recall: } \frac{\partial}{\partial x}(e^{ikx}) = ike^{ikx}}$ |Replace (1) by:

$\frac{\partial u}{\partial t} + c\delta_{2x}u = 0$ (2) $\Rightarrow \frac{\partial u}{\partial t} + ci\left(\frac{\sin k\Delta x}{\Delta x}\right)e^{ikx} = 0$ |Since new form find soln to match:

Assume a soln of the form $u(x,t) = A(t)e^{ikx}$ (3)

Subst (3) into (2) $\Rightarrow \frac{dA}{dt}e^{ikx} + cA(t)\delta_{2x}(e^{ikx}) = 0$

$\frac{dA}{dt}e^{ikx} + ic\left(\frac{\sin k\Delta x}{\Delta x}\right)e^{ikx}A(t) = 0 \Rightarrow \frac{dA}{dt}\cancel{e^{ikx}} + ic\left(\frac{\sin k\Delta x}{\Delta x}\right)\cancel{e^{ikx}}A(t) = 0$

$\left.\begin{array}{l} \frac{dA}{dt} = i\left(-c\frac{\sin k\Delta x}{\Delta x}\right)A \\ A(0) = A_0 \end{array}\right\}(4)$ | This is osc eqn with $\omega = -c\frac{\sin k\Delta x}{\Delta x}$ where ω is a constant

$\therefore$ we can say previous results apply

$\therefore$ Soln to $(4) = A(t) = A_0 e^{i\omega t} \Rightarrow u(x,t) = A_0 e^{i\omega t}e^{ikx}$ (5)

$\Rightarrow u(x,t) = A_0 e^{i((kx+\omega t))} \Rightarrow \boxed{u(x,t) = A_0 e^{ik\left(x+\frac{\omega}{k}t\right)}}$ |aprox soln , space deriv 48 min

$\Rightarrow A_0 \exp\left[ik\left(x - c\frac{\sin k\Delta x}{\Delta x}t\right)\right]$ | notice this is similar to exact sol except we have c' term

$\Rightarrow A_0 e^{ik(x-c't)}$ | where $c' = c\frac{\sin k\Delta x}{\Delta x}$ travelling wave w/ phase speed c'

Amp=A_0, Travellingwave, $\lambda = \frac{2\pi}{k}$, phase speed=$c\frac{\sin k\Delta x}{\Delta x}$

form: $u(x,t) = A(t)e^{ikx} \Rightarrow \frac{dA}{dt} = i\omega A$ (2)

leads to ODE, happens to be the Osc eqn w/ a particular form of ω

$\omega = -c\frac{\sin k\Delta x}{\Delta x}$ (3) previous soln: $\boxed{A_{n+1} = A_n(1+i\omega\Delta t)}$

Stability Condition for LF

$\omega = -c\frac{\sin k\Delta x}{\Delta x}$ $\left(\frac{|c||\sin k\Delta x|}{\Delta x}\right)\Delta t < 1 \Rightarrow \frac{|c|\Delta t}{\Delta x}|\sin k\Delta x| < 1$

LF time, 2nd order space: $\boxed{u_j^{n+1} = u_j^{n-1} - \frac{c\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n)}$

Numerical Modeling II

$e^{ix}-e^{-ix}=2i\sin x$, $e^{ix}+e^{-ix}=2\cos x$, $e^{i\pi}=-1$, $e^{i\theta}=\cos\theta+i\sin\theta$

$\frac{1}{i}=-i$, $\delta_x^2 u_j=\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta^2 x}$

γ is a smoothing parameter. $\gamma_{2l}=\left(\frac{\Delta x^2}{4}\right)^l$

$$\delta_x^2\left(e^{ikx}\right)=\frac{1}{\Delta x^2}\left(e^{ik(j+1)\Delta x}-e^{ik(j)\Delta x}-e^{ik(j)\Delta x}+e^{ik(j-1)\Delta x}\right)\to\frac{dA}{dt}=-2\gamma_2\left(\frac{1-\cos(k\Delta x)}{\Delta x^2}\right)A$$

$$\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+c\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}=\upsilon\frac{u_{j+1}^{n-1}-2u_j^{n-1}+u_{j-1}^{n-1}}{\Delta x^2} \quad \text{or} \quad \delta_{2t}u_j^n+c\delta_{2x}u_j^n=\upsilon\delta_x^2u_j^{n-1}$$

When $c=0$, $\frac{u_j^{n+1}-u_j^{n-1}}{\Delta t}=-\gamma_4\delta_x^4u_j^n$

(1) $\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-g\frac{\partial h}{\partial x}$ | $-g\frac{\partial h}{\partial x}=PGF$ term; U is constant

(2) $\frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}=-H\frac{\partial u}{\partial x}$ | $-H\frac{\partial u}{\partial x}=$ Divergence term

$$\left.\begin{array}{l}(1)\ \frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=-g\frac{\partial h}{\partial x}\\(2)\ \frac{\partial h}{\partial t}+U\frac{\partial h}{\partial x}=-H\frac{\partial u}{\partial x}\to\begin{cases}u=ae^{ik(x-ct)}\\h=be^{ik(x-ct)}\end{cases}\end{array}\right\}\to c=U\pm\sqrt{gH}\ \text{|Spceial case}$$

$$(\mathrm{PGF})_j^n=-g\frac{h_{j+1/2}^n-h_{j-1/2}^n}{\Delta x}=-g\delta_xh_j^n;\quad\left(\begin{array}{c}Div\\term\end{array}\right)_{j+1/2}^n=-H\frac{u_{j+1}^n-u_j^n}{\Delta x}=-H\delta_xu_{j+1/2}^n$$

$\delta_{2t}u_j^n+U\delta_{2x}u_j^n=-g\delta_xh_j^n;$

$\delta_{2t}h_{j+1/2}+U\delta_{2x}h=-H\delta_xu_{j+1/2}^n$

$$Tr\equiv\text{Approx} - \text{Exact}\Rightarrow Tr\equiv\frac{h\left(x+\frac{\Delta x}{2}\right)-h\left(x-\frac{\Delta x}{2}\right)}{\Delta x}-\frac{\partial h}{\partial x}$$

Expand $h\left(x+\frac{\Delta x}{2}\right)$ & $h\left(x-\frac{\Delta x}{2}\right)$ with Taylor series:

$$h\left(x+\frac{\Delta x}{2}\right)=h(x)+\frac{h'(x)\frac{\Delta x}{2}}{1!}+\frac{h''(x)\left(\frac{\Delta x}{2}\right)^2}{2!}+\frac{h'''(x)\left(\frac{\Delta x}{2}\right)^3}{3!}+\ldots$$

$$h\left(x-\frac{\Delta x}{2}\right)=h(x)-\frac{h'(x)\frac{\Delta x}{2}}{1!}+\frac{h''(x)\left(\frac{\Delta x}{2}\right)^2}{2!}-\frac{h'''(x)\left(\frac{\Delta x}{2}\right)^3}{3!}+\ldots$$

$$Tr\equiv\frac{\frac{2\frac{\partial h}{\partial x}\frac{\Delta x}{2}}{1!}+2\frac{\frac{\partial h^3}{\partial x^3}\left(\frac{\Delta x}{2}\right)^3}{3!}}{\Delta x}-\frac{\partial h}{\partial x}\Rightarrow Tr\equiv\frac{\partial h^3}{\partial x^3}\left(\frac{\Delta x^2}{24}\right)$$

Time-diff	Advection	Diffusion
Forward	U	S
Leapfrog	S	U

$S_{2\pi}=\{f:f(x+2\pi)=f(x)\text{ all }x\}=$ definition of a 2π periodic function

Basis - Definition : The vectors $e_1,e_2,\ldots,e_n$ form a basis for V if each vector in V can be uniquely expressed as a linear combination of the vectors $e_1,e_2,\ldots,e_n$

Basis of $S_{2\pi}=\{e^{ikx},\ k=0,\pm1,\pm2,\ldots\}$ If $f\in S_{2\pi},f(x)=\sum_{k=-\infty}^{\infty}a_ke^{ikx}$, {this is fourier series}

For f to be real-valued: $a_{-k}=a_k^*$

$a_ke^{ikx}+a_{-k}e^{-ikx}=a_ke^{ikx}+\left(a_ke^{ikx}\right)^*=2\,\mathrm{Re}\left[a_ke^{ikx}\right]$

$f(x)=a_0+\left[\sum_{k=1}^{\infty}a_ke^{ikx}+cc.\right]$, where $cc=$ complex conj

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Spectral Method AKA projection method, (here applied to linear advection eqn)

$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,\qquad u(x,t)=\sum_{k=-\infty}^{\infty}a_k(t)e^{ikx}$

$\frac{\partial u}{\partial t}=\sum_{k=-\infty}^{\infty}\frac{da_k}{dt}e^{ikx},\qquad\frac{\partial u}{\partial x}=\sum_{k=-\infty}^{\infty}ika_k(t)e^{ikx}$

$$\therefore\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\sum_{k=-\infty}^{\infty}\frac{da_k}{dt}e^{ikx}+\sum_{k=-\infty}^{\infty}icka_k(t)e^{ikx}\Rightarrow\sum_{k=-\infty}^{\infty}\left[\frac{da_k}{dt}+ikca_k(t)\right]e^{ikx}=0$$

IC: $f(x)=u(x,0)=\sum_{k=-\infty}^{\infty}a_k(0)e^{ikx}$, Assume: $u(x,t)=\sum_{k=-\infty}^{\infty}a_ke^{ikx}$ {ipod ~20 m}

$\Rightarrow\sum_{k=-\infty}^{\infty}\left[\frac{da_k}{dt}+ikca_k(t)\right]e^{ikx}=0$ {Adv eqn from before}

Require: $<\left[\frac{da_k}{dt}+ikca_k(t)\right]e^{ikx},\ e^{ilx}>\ =0$

$=0\left[\frac{da_k}{dt}+ikca_k(t)\right]=0\quad\{\text{when }k\neq l\}$

$=2\pi\left[\frac{da_k}{dt}+ikca_k(t)\right]=0\quad\{\text{when }k=l\}$

$\frac{da_k}{dt}+ikca_k(t)=0$ {now have ODE} $\to\frac{da_k}{dt}=-ikca_k,$

$\frac{da_k}{a_k}=-ikcdt\to\int\frac{1}{a_k}da_k=\int-ikcdt\to\ln a_k=-ikct\to$

$e^{\ln a_k}=e^{-ikct}\to a_k(t)=a_k(0)e^{-ikct}$

II) Do inner product: $<u(x,0),e^{ikx}>\ =\ <a_k(0)e^{-ikct},e^{ikx}>$

$=\ a_k(0)=\frac{1}{2\pi}\int_0^{2\pi}f(x)e^{-ikx}dx$

Spectral Methods

The Spectrum of a Function

- Integral transforms and the spectrum of a function are closely related; in fact, an integral transformation can be thought of as a resolution of a function into a certain spectrum of components (Farlow 1982, p. 74).
- Although the time step is more restricted with the spectral method than with centered differences, the solution is more accurate for a given wavenumber and fewer waves need to be retained in the solution for comparable accuracy, leading to a coarser grid and larger Δx (Mote 2000)
- The spectral method does not introduce phase speed or amplitude errors, even in the shortest wavelengths (Duran 1999, p. 178)

Finite Fourier Transform

- If the Fourier-series expansion of a real-valued function is truncated at wave number N, the set of Fourier coefficients contains $2N+1$ pieces of data.

Classic Initial - Boundary - Value Problem

$$\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial x^2}, \; 0 < x < \pi \qquad (1) \text{ Diffusion Eqn}$$

$$u(0,t) = u(\pi,t) = 0, \; t > 0 \qquad \text{Boundary conditions}$$

$$u(x,0) = f(x) \qquad \text{Initial condition}$$

Theorem

- Any function satisfying the B.C. can be expressed as an infinite series in the functions $\sin(nx), \; n = 1,2,...$
- The set $\{\sin(nx)\}, n = 1,2,...$ is a basis for the vector space of functions that satisfy the B.C.

Orthogonality

$$\int_0^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 \text{ if } m \neq n \\ \pi/2 \text{ if } m = n \end{cases}$$

Series Solution

- Write solution to (1) in the form $u(x,t) = \sum_{m=1}^{\infty} a_m(t)\sin(mx)$ & substitute into (1)
- This function automatically satisfies the BC
- Now need to satisfy: PDE, IC

Part 1 : NUMERICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS

Chapter 1 : Finite difference methods

in 1-D, time is our dependent variable (in ODE)

Recall: $f'(t) = \lim_{\Delta t \to 0} \dfrac{f(t+\Delta t) - f(t)}{\Delta t}$ for small Δt, $\boxed{f'(t) \cong \dfrac{f(t+\Delta t) - f(t)}{\Delta t}}$ | As $\Delta t \downarrow$, cost $\uparrow$

Forward diff. If $\Delta t > 0$, we are loking forward in time. Marching forward in time steps:

|– – –|– – –|– – –|– – –|– – –|– – –|– – – >

0	Δt	$2\Delta t$	$3\Delta t$	$n\Delta t$	t
t_0	t_1	t_2	t_3	t_n	

$\Delta t = t_1$, $\therefore$ $\boxed{t_n = n\Delta t}$, where Δt = time step

Notation :

$Y_{ex}(t)$ = exact solution to ODE

$\boxed{Y_n(t) \equiv Y_{approx}(n\Delta t)}$ = approximate solution at time (t_n)

Example 1 : General ODE

(1) $\dfrac{dy}{dt} = f(y)$

- A unique solution is possible if an initial condition is given, e.g. $y(0) = y_0$
- Simplest method for solving it is (2):

(2) $\boxed{\left(\dfrac{dy}{dt}\right)_n \cong \dfrac{Y_{n+1} - Y_n}{\Delta t}}$ | If $\Delta t > 0$ then it is: Forward Difference Solution.

(3) $\dfrac{Y_{n+1} - Y_n}{\Delta t} = f(y_n) \Rightarrow \boxed{Y_{n+1} = Y_n + \Delta t\ f(y_n)}$

- **Euler's Method** (Forward time differencing)
- An approximation for (1) made by replacing LHS of (1) with RHS of (2) and setting it equal to $f(y_n)$

Example 2 : IVP ("Decay/friction" equation)

(4) $\dfrac{dy}{dt} = -Ky$, $Y(0) = (Y_0)$

$f(y) = -Ky$, $K > 0$, Exact solution: $\boxed{Y_{ex}(t) = y_0 e^{-Kt}}$

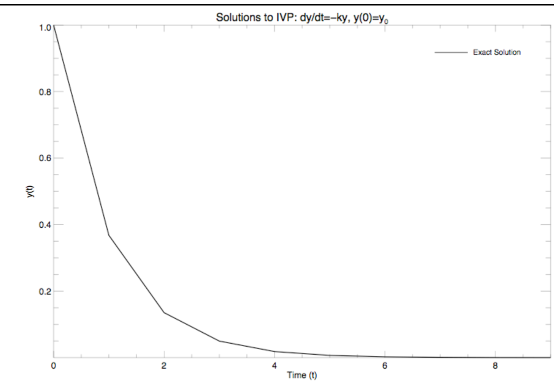
Solution is monotonically decreasing. Properties:

- as $t \to \infty$, $y_{ex}(t) \to 0$.

  if $t_1 < t_2 \Rightarrow y_{ex}(t_2) < y_{ex}(t_1)$
- $\lim_{t \to 0} y_{ex}(t) = 0$

  Note: $y_{ex}(t)$ does not change sign

In finding approximate solutions, we want these properties to be preserved.



• To find approximate solution, apply Euler's method to (4) & solve for y_n :

$$\frac{y_{n+1}-y_n}{\Delta t} = -ky_n \;\Rightarrow \boxed{y_{n+1} = (1-k\Delta t)\,y_n}$$

| where k has to have dimensions of inverse time (since it is subtracted from 1)

$y_1 = (1-k\Delta t)\,y_0$,

$y_2 = (1-k\Delta t)\,y_1 = (1-k\Delta t)^2\,y_0$

(5) $\boxed{y_n = (1-k\Delta t)^n\,y_0}$

• Goal/hope: as $\Delta t \to 0$, approx soln $\to$ exact soln; *i.e.* $y_n \to y_{ex}$

In general, it is very hard know to this, if we do not have analytical soln.

In the above case, we can do that directly.

$\boxed{y_{ex}(t) = y_0 e^{-Kt}}$ $\Leftarrow$ Recall: exact solution to (4)

Substitute $t_n = n\Delta t$ into (4): $y_{ex}(n\Delta t) = y_0 e^{-kn\Delta t} \Rightarrow \boxed{Y_{ex}(n\Delta t) = y_0\left(e^{-K\Delta t}\right)^n}$

Notice $(1-k\Delta t)^n = \left(e^{-K\Delta t}\right)^n$ and $e^{-k\Delta t} = 1 - k\Delta t + \frac{(k\Delta t)^2}{2} + \ldots$ | 1st 2 terms of Taylor exp..

Note: this not an exact solution. Another method is to do it graphically, then we see that the approx sln converges to exact soln.

The behavior of the solution depends on the value of $k\Delta t$, such that $y_n = (1-k\Delta t)^n\,y_0$

Case 1: $0 > k\Delta t < 1$; Properties:

- Because of the power rel. function gets smaller ST $y_{n+1} < y_n$ for all n *i.e.* monotonically decreasing function.
- "If you take a fraction to the nth power it is zero" $\lim_{n\to\infty} y_n = 0$

$\therefore$ Two basic properties from exact solution are preserved

Case 1A: $k\Delta t = 1$; $y_n = 0$ @ $n \ge 1$

Case 2: $1 < k\Delta t < 2$

$-1 < 1 - k\Delta t < 0 \;\Rightarrow\; |1-k\Delta t| < 1$

$\boxed{y_n = (1-k\Delta t)^n\,y_0} \;\Rightarrow\; |y_n| \to 0 \;\; as\; n \to \infty$

Note: we get negative values! Solution does approach 0, but it is not monotonic.

Bifurcation: It is changing its behavior (when the sign changes)

Case 3 $k\Delta t > 2$

$\Rightarrow 1 - k\Delta t < -1 \Rightarrow |1 - k\Delta t| > 1$

$|y_n| \to \infty$ as $n \to \infty$. $\therefore |y_{n+1}| > |y_n|$

Properties:

• Unstable solution: blows up

If Δt is too large, it behaves badly.

This is an example of **Numerical Instability**

If exact soln is bounded but the approx soln is unbounded

ie. approx soln $\to 0$ as $n \to 0$

Mon, Jan 29

The Oscillation Equation $\begin{cases} \dfrac{dy}{dt} = i\omega y \\ y(0) = y_0 \end{cases}$

Solution : $\boxed{y = y_0 e^{i\omega ty}}$ or $\Rightarrow \boxed{y = y_0\left(\cos\omega t + i\sin\omega t\right)}$ $\{$See Emerson Fig. 1$\}$

- Assume y_0 is real. (take real part for solutions, generally ~~check)
- $\mathrm{Re}(y) = y_0 \cos\omega t$, where ω [rad s^{-1}] = angular frequency. Can be + or −
- Period : $\dfrac{2\pi}{|\omega|} = P$; *ie*. period $= \dfrac{1}{f}$

2 aspects to an Oscillation (you always want to get right in a numerical scheme):

- Period: P
- Amplitude: y_0

Oscillation Eqn (Duran, p. 49):

$(2.30)\ \boxed{\dfrac{d\psi}{dt} = i\kappa\psi}$, where κ = frequency (real constant). Integrating (2.30) over a time Δt yields:

$(2.31)\ \boxed{\psi(t_0 + \Delta t) = e^{i\kappa\Delta t}\psi(t_0) \equiv A_e\psi(t_0)}$, where $A_e \equiv$ "exact amplification factor", a complex number of modulus one. According to (2.31), ψ moves $\kappa\Delta t$ radians around a circle of radius $|\psi(t_0)|$ in the complex plane over the time interval Δt.

~~~~~~

DIFFERENCING SCHEMES :

[1] Euler, [2A] Implicit, [2B] Trapezoidal Implicit, [4] Leapfrog

[1] Euler : $\dfrac{y_{n+1} - y_n}{\Delta t} = i\omega y_n$ • Want to focus on the real part

$\Rightarrow y_{n+1} = (1 + i\omega\Delta t)y_n \quad \Rightarrow \quad y_n = (1 + i\omega\Delta t)^n y_0$

We want the absolute value to see if it blows up: $|y_n| = \left|(1 + i\omega\Delta t)^n\right||y_0|$

Note: $\boxed{|x + iy| = \sqrt{x^2 + y^2}}$ $\Rightarrow$ $\boxed{|1 + i\omega\Delta t| = \sqrt{1 + \omega^2\Delta t^2}}$ | $\omega^2\Delta t^2$ is always > 0

- $i\omega\Delta t > 1$ then grows
- $i\omega\Delta t < 1$ then shrinks

Conclude: > 1 for all t

- both are > 1 for all $t > 0$; $\{$See Fig. 2$\}$
~~~~~~

Conclusion:

- Euler's method is UNSTABLE for osc eqn. and is so for all values of $|\omega|\Delta t > 0$
  ie. unconditionally unstable, no matter how small you make Δt, it is unstable, though
  a smaller Δt takes longer to blow up. Here diff scheme
  is producing the instability. ie. NUMERICAL INSTABLITY, created by mathematiical
  technique, does not represent anything in nature.
  You do not want the diff scheme to create the unstability.
- EULERS METHOD WILL NOT WORK FOR ANY WAVE EQNS

2 IMPLICIT METHOD / SCHEME (somewhat of a misnomer)

Recall Osc Eqn: $\frac{dy}{dt} = i\omega y$. Rewrite LHS $\Rightarrow \boxed{\frac{y_{n+1} - y_n}{\Delta t} = i\omega y_{n+1}}$ (A)

NB. In generally, we are trying to solve eqns of this form: $\boxed{\frac{dy}{dt} = f(y)}$,

and we take $\frac{y_{n+1} - y_n}{\Delta t} = f(y_{n+1})$, solve this as: $y_{n+1} = y_n + \Delta t f\left(y_{n+1}\right)$

$\therefore$ since there is an unknown on both sides, this is known as Implicit Method

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COMPARE

1 Euler's method

- $\boxed{y_{n+1} = y_n + \Delta t\ f\left(y_n\right)}$
- Explicite solution: All is known on the RHS

2 Implicit Method

- $\boxed{y_{n+1} = y_n + \Delta t\ f\left(y_{n+1}\right)}$
- Implicit: computationally more expensive

$\therefore$ Since Implicit is so expensive, **semi - implicit** is sometimes used. This method splits the RHS into liner & nonliner parts

(A) above can be rewritten as: $y_{n+1} = y_n + \left(i\omega\Delta t\right) y_{n+1} \Rightarrow\ y_n = y_{n+1} - \left(i\omega\Delta t\right) y_{n+1} \Rightarrow$

$y_n = y_{n+1}\left(1 - i\omega\Delta t\right) \Rightarrow \boxed{y_{n+1} = \left(\frac{1}{1 - i\omega\Delta t}\right) y_n}$, where $\lambda \equiv \frac{1}{1 - i\omega\Delta t}$

Solution: $\boxed{y_n = \lambda^n y_0}$ ABSOLUTE VALUE OF λ IS CRITICAL! IF:

- $\lambda < 1 \Rightarrow$ damped, shrinks
- $\lambda = 1 \Rightarrow$ steady, stays the same
- $\lambda > 1 \Rightarrow$ grows

$\boxed{|\lambda| = \frac{1}{|1 - i\omega\Delta t|} = \frac{1}{\sqrt{1 + \omega^2\Delta t^2}}\ \&\ \therefore\ \text{always} < 1}$ & $\therefore$ sol decays exponentially

{See Emerson Fig 3}

$\boxed{\lambda: \text{ whether it grows or is dissipates depends on value of } \lambda}$

• Case: $\lambda < 1 \Rightarrow$ **Dissipative scheme**

$\rightarrow$ this is artificial damping, dissipation (of the amplitude). It is as if the differencing scheme has its own inherent friction/damping effect.

$\rightarrow$ exact solution to the ODE is not damped, but computed solution is, $\therefore$ it is still not correct.

$\rightarrow$ If damping is small and does not go too long, it may be useful. Selective damping is a good thing (ie small, sound). Some times they add dissipative terms to dampen selective waves . This can be used to dampen poorly resolved waves, i.e. resoved, but poorly resolved (e.g. at lower end of grid scale). Δt needs to be chosen carefully.

$\rightarrow$ Implicit is stable, need to focus on accuracy. Need to pick Δt for accuracy

$\rightarrow$ Use semi-implict to be able to increase tme step. Above is just one example of an Implict.

2B Trapezoidal Implicit {See Emerson fig 6}

• Average the input differences b/t Euler's y_n & Implicit y_{n+1}

Generally: $\frac{dy}{dt} = f(y), \quad \frac{y_{n+1} - y_n}{\Delta t} = \frac{i\omega}{2}\left[f(y_n) + f(y_{n+1})\right]$

Solution: "A form of it"

$\boxed{y_n = \lambda^n y_0}$ with $|\lambda| = 1 \Rightarrow |y_n| = |y_0|$ {See Emerson fig 7}

• The amplitude is exactly correct, but the period might be off. This would lead to the phase error $\uparrow$

• When trapezoidal is referred to, this is usually this method. Very popular in certain...

Wed, Jan 31

Review: Complex numbers:

- $x+iy=re^{i\theta} \Rightarrow x=$ real part, $y=$ imaginary part.
- one complex # $=2$ real equations

$\boxed{x+iy=re^{i\theta}}$, $\boxed{r=\sqrt{x^2+y^2}}$, $\boxed{\theta=\tan^{-1}\left(\frac{y}{x}\right)=\sin^{-1}\left(\frac{y}{r}\right)}$ {See Emerson fig 1.}

Raise complex # to a power:

- $(x+iy)^n \Rightarrow$ (problematic) it has n terms in it, odd $\Rightarrow$ complex, even $\Rightarrow$ real
- polar coordinates (simple) $\Rightarrow$ $\boxed{(x+iy)^n=\left(re^{i\theta}\right)^n=r^n e^{in\theta}=r^n\left(\cos n\theta+i\sin n\theta\right)}$

TRAPEZOIDAL SCHEME: solving the oscillation equation

- Implicit method
- uses forward differencing

$$\frac{dy}{dt}=i\omega y,\ \ y(0)=y_0$$

$\frac{y_{n+1}-y_n}{\Delta t}=i\omega\left(\frac{y_n+y_{n+1}}{2}\right)$, solve for $y_n=\lambda^n y_0$, now $\lambda=\frac{1+\frac{i\omega\Delta t}{2}}{1-\frac{i\omega\Delta t}{2}}$, | $|\lambda|=1$ b/c $1=$ |num & denom|

Now put this in polar coordinates such that $\lambda=e^{2i\alpha}$ where $\alpha=\tan^{-1}\frac{\omega\Delta t}{2}$ {See EL Fig 2}

- Nice to be able to plot up one of these things. Nice to have a compact formula.
- "Try to do as much algebra as possible before performing computations;

(*) $\boxed{y_n = \left(e^{i2\alpha}\right)^n y_0}$ Approximate

- We know that the approximate solution has the right amplitude b/c $|\lambda| = 1$ but may not have the right frequency. What is the approx freq & how does it compare to exact? Technique: write it in the same form as exact soln.

  $y_{ex}(n\Delta t) = e^{i\omega n\Delta t} y_0 \Rightarrow$

(**) $\boxed{y_{ex}(n\Delta t) = \left(e^{i\omega\Delta t}\right)^n y_0}$

- Exact solution. $P = \dfrac{2\pi}{|\omega|}$; $f = \dfrac{|\omega|}{2\pi}$; $\omega = \omega$
- What is the angular frequency (ω_T) of the calculated solution y_n?

  Look at how freq appears in exact solution: $i\omega\Delta t$ Approx: $i2\alpha$
- Force approx argument to have the same as exact

$\boxed{\omega_T\left[\text{rad s}^{-1}\right] = \dfrac{2\alpha}{\Delta t}}$, where $\alpha = \tan^{-1}\left(\dfrac{\omega\Delta t}{2}\right) \Rightarrow$ Full form: $\boxed{\omega_T = \dfrac{2\tan^{-1}\left(\dfrac{\omega\Delta t}{2}\right)}{\Delta t}}$

- Is this dimensionally correct? $\omega\left[\text{s}^{-1}\right]$ & $\Delta t[\text{s}]$ $\therefore$ looks OK dimensionally
- look at $\dfrac{\omega_T}{\omega}$ ratio: $\boxed{\dfrac{\omega_T}{\omega} = \dfrac{\dfrac{2\tan^{-1}\left(\dfrac{\omega\Delta t}{2}\right)}{\Delta t}}{\omega}} \Rightarrow \boxed{\dfrac{\omega_T}{\omega} = \dfrac{\tan^{-1}\left(\dfrac{\omega\Delta t}{2}\right)}{\dfrac{\omega\Delta t}{2}}}$ starting point for HW{~30min}

- HW: plot this as a function of $\omega\Delta t$, see what it looks like: Hint: @ 0 will get error, but if we expand in taylor series, ratio appproaches 1 as $\omega\Delta t \to 0$
- we will prove: $\dfrac{\omega_T}{\omega_{ex}} < 1 \Rightarrow \boxed{\omega_T < \omega_{ex}} \Rightarrow \boxed{P_T > P_{ex}}$ {See figs 3 & 4}

$$\left.\begin{aligned} P_T &= \frac{2\pi}{\omega_T} \\ P_{ex} &= \frac{2\pi}{\omega} \end{aligned}\right\}\text{Periods}$$

- Functions get more & more out of phase, eventually they will be 1/2 period out of phase, then go back in etc.....
- This differencing scheme gives exact ampl, but period is wrong.

HW2 discussion: see HW 2 sheet

Aliasing: See fig 5

Assumption: $\Delta t = 1$ period

"Sampling the oscilation, ex. 8000 hz

* Shortest resolvable period = $2\Delta t$

Ex: $p = \Delta t$

• no point looking at $2\Delta t$

General Rule:

If $p_1 < 2\Delta t$, then there exists $p_2 > 2\Delta t$ ST the 2 oscillations are indistinquishable

•There is always a lower limit to period.

• If there is smallest period there is a largest frequency: you can not get above that freq.

Interqactive Aliasing tool:

www.dsptutor.freeuk.com/aliasing/AD102.html

ex: if freq is too high, then there are 2 diff functions that pass through same point

if input freq is 3000HZ it IS RESOLVABLE

$2\Delta t$ is resovalble, but jagged. Thee biiigets delt could be is half of the period.

undersampling vs oversampling

SUMMARY:

• $P_{\min} = 2\Delta t \Rightarrow \omega_{\max} = \frac{2\pi}{p_{\min}} = \frac{\pi}{\Delta t}$

• $\boxed{\omega_{\max}\Delta t = \pi}$ HW2b

• All methods so far have been 2-level methods.

Next time:

LEAPFROG: $\omega_T > \omega$

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Monday, Feb 5

Implicit methods - Complications (Preamble to leapfrog method)

• Solving System of linear eqns w/ Implicit scheme

Explicit Scheme :

$$\left.\begin{array}{l}\dfrac{dx}{dt}=ax+by\\ \dfrac{dy}{dt}=cx+dy\end{array}\right\}\Rightarrow\begin{cases}\dfrac{x_{n+1}-x_n}{\Delta t}=ax_n+by_n\Rightarrow x_{n+1}=x_n+\Delta t\left(ax_n+by_n\right)\\ \dfrac{y_{n+1}-y_n}{\Delta t}=cx_n+dy_n\Rightarrow y_{n+1}=y_n+\Delta t\left(cx_n+dy_n\right)\end{cases}$$

Implicit Scheme :

$$\left.\begin{array}{l}\dfrac{dx}{dt}=ax+by\\ \dfrac{dy}{dt}=cx+dy\end{array}\right\}\Rightarrow\begin{cases}\dfrac{x_{n+1}-x_n}{\Delta t}=ax_{n+1}+by_{n+1}\Rightarrow\ (1-a\Delta t)x_{n+1}-(b\Delta t)y_{n+1}=x_n\\ \dfrac{y_{n+1}-y_n}{\Delta t}=cx_{n+1}+dy_{n+1}\Rightarrow\ (1-d\Delta t)y_{n+1}-(c\Delta t)x_{n+1}=y_n\end{cases}$$

$\therefore$ we will get a system of linear eqns to solve at every grid point. In some cases it is worth it. If we had system of non-linear eqns, we had have to iterate each time step before going on to next time step. Conclusion: Implicit is not worth it if it is compuationally too expensive.

ACCURACY

• So far: all discussion has been re. forward differencing: $\left(\dfrac{dy}{dt}\right)_n\cong\dfrac{y_{n+1}-y_n}{\Delta t}$ | RHS is forward diff

See Fig 1.

Forward Diff:

Centered Difference: $\boxed{f'(t_0)\simeq\dfrac{f(t_0+\Delta t)-f(t_0-\Delta t)}{2\Delta t}}$

Hypothesis: Centered is more accurate than forward diff

Local truncation error: $T_R\equiv$ approx expression minus exact

$$T_R(F)=\frac{f(t+\Delta t)-f(t)}{\Delta t}-f'(t)$$

$$T_R(C)=\frac{f(t+\Delta t)-f(t-\Delta t)}{2\Delta t}-f'(t)$$

• Key in comparing: Assume Δt is small.

• Approach: expand in power series & then compare expansions.

$a_1x+a_2x^2+a_3x^3+\ldots\{\textit{fillin}$ from emerson$\}$

Let $x\to0$

$\lim\limits_{x\to0}\ -\{\textit{fillin}$ from emerson$\}$

Summ : higher order term approaches zero faster than lower order.

General: if $n > m$ $\boxed{\lim_{x\to 0}\frac{a_n x^n}{a_m x^m} = 0}$

- $a_0 x + a_1 x^1 + a_2 x^2 \cong a_0 x$ for small x ie. leading term that determines behavior

Now, do Taylor series expansion on above:

$$T_R(F) = \frac{f'(t)\Delta t + \frac{f''(t)}{2}\Delta t^2 + \ldots}{\Delta t} - f'(t)$$

$$= \cancel{f'(t)} + \frac{f''(t)}{2}\Delta t + \ldots. \quad - \cancel{f'(t)} \qquad \Rightarrow \boxed{\frac{f''(t)}{2}\Delta t + \ \ldots \text{(higher order terms)}}$$

$T_C(F) =$ fill in emerson

$$\boxed{T_C(F) = \frac{f'''(t)}{3!}\Delta t^2 + \ldots \text{(higher order terms)}}$$

Conclusion:

$T_R(F)$: First order accurate
$T_C(F)$: Second order accurate: more accurate than 1st order

Next time: leap frog

Wed. Feb 7, Truncation Errror & Leap Frog

$Tr = O(\Delta t^k)$; | Em: The leading term in Taylor expansion is α to Δt^k
"Don't go beyond 2nd, price is too high"

- the larger $k \Rightarrow$ the more accuurate

Centered Time Diffferencing

$$\left.\begin{array}{l}\frac{dy}{dt} = f(y) \\ y(0) = y_0\end{array}\right\} \text{IVP}$$

$$\left(\frac{dy}{dt}\right)_n \cong \frac{y_{n+1} - y_{n-1}}{2\Delta t} \quad \Rightarrow \quad \frac{y_{n+1} - y_{n-1}}{2\Delta t} = f(y_n) \Rightarrow \boxed{y_{n+1} = y_{n-1} + 2\Delta t \ f(y_n)} (*)$$

Leapfrog Method : going from back to front using the person in the middle

- 2nd order accurate
- explicit method $\Rightarrow$ easy to program

To get it going, need to specify: y_0 and y_1

Drawbacks :

- NOT an IVP (in the strict sense) since we need to specify 2 values

Procedures to get y_1 :

- Use Euler's Method: $y_1 = y_0 + \Delta t\ f(y_0) \Rightarrow$

  $y_{n+1} = y_{n-1} + 2\Delta t f(y_n)$ for $n = 1,2,...$

  – Even though Euler's method is unstable for the osc eqn, can be used for 1 time step w/out degrading the accuracy

- Set $y_1 = y_0$. Note: not a good approach

Example : Apply Leapfrog to osc eqn

- y_0: specified in IVP
- $y_1 = (1 + i\omega\Delta t)y_0$
- Then, apply LF from here: $y_{n+1} = y_{n-1} + (2i\omega\Delta t)(y_n)$, $n = 1,2,...$ $(**)$
- Look for solutions of the form $y_n = r^n$, where r is some # TBD
- Subst. this into $(**)$ to see if there are solutions of this form: $\boxed{r^{n+1} = r^{n-1} + (2i\omega\Delta t)(r^n)}$
- divide by lowest power, i.e. r^{n-1}, $\Rightarrow r^2 = 1 + (2i\omega\Delta t)r \Rightarrow \boxed{r^2 - (2i\omega\Delta t)r - 1 = 0}$
- Quadratic formula produces 2 solutions: $\boxed{r_\pm = i\omega\Delta t \pm \sqrt{1 - \omega^2\Delta t^2}}$
- General Solution to $(**)$ is a linear combination: $\boxed{y_n = ar_+^n + br_-^n}$ (~ 36min)

  where a & $b \equiv$ arbitrary constants determined using/from y_0 & y_1

  If you plug any expression of that form into $(**)$ then it's solved.

  We have two variables to determine, but we have specified 2 things;

  y_0 & y_1 are specified, and a and b are determined by that specification.

  HW notes: you plug in: • $n = 0$, get $y_0 =$ _____

  • $n = 1$, then solve for a & b

Case 1 : $\omega\Delta t > 1$

$$r_\pm = i\omega\Delta t \pm \sqrt{(-1)(\omega^2\Delta t^2 - 1)} \Rightarrow r_\pm = i\omega\Delta t \pm i\sqrt{(\omega^2\Delta t^2 - 1)} \Rightarrow$$

$r_\pm = i\left(\omega\Delta t \pm \sqrt{(\omega^2\Delta t^2 - 1)}\right)$ |Both #s are on the Img axis. Since $|i| = 1$, take Abs value $\Rightarrow$

$$|r_\pm| = \left|\omega\Delta t \pm \sqrt{(\omega^2\Delta t^2 - 1)}\right|.$$

If either r_+ or r_- has an abs val > 1, then y_n blows up. 1st look at the r_+ case:

$\rightarrow$ for stable sol abs val of both roots can not be GT 1

$|r_+| = \left|\omega\Delta t \pm \sqrt{(\omega^2\Delta t^2 - 1)}\right|$ Since both terms on RHS are positive, then:

$$|r_+| = \left|\omega\Delta t + \sqrt{(\omega^2\Delta t^2 - 1)}\right| = |r_+| = \omega\Delta t \pm \sqrt{(\omega^2\Delta t^2 - 1)}$$

Key question: Is this > 1? Since it is, then $|r_+| > 1 \Rightarrow$ solution grows exponentially

$\therefore$ Method is unstable in this case, (conditinionally unstable if othher cases are stable)

Summary: $\boxed{\text{If } \omega\Delta t > 1, \text{ Leapfrog (applied to osc eqn) is numerically unstable}}$

$\therefore$ for given ω, Δt can not be too big. $\therefore$ Try smalller values of Δt

Case 2 : $\omega\Delta t < 1$, $\quad r_\pm = i\omega\Delta t \pm \sqrt{1-\omega^2\Delta t^2}$ (term under rad is Real b/c not negative)

• Try squaring both sides: $\quad |r_\pm|^2 = \omega^2\Delta t^2 + \left(1-\omega^2\Delta t^2\right) \Rightarrow \boxed{|r_\pm|^2 = 1}$

This happpens because it is bounded: $y_n = ar_+^n + br_-^n \quad \Rightarrow$

ie $@r = 1$ $\;|y_n| \le |a| + |b|$; $\quad |a|+|b|$ will never be bigger than __________

Summary: $\boxed{\text{If } \omega\Delta t < 1, \text{ Leapfrog (applied to osc eqn) is numerically stable}}$

$$a = \frac{1+\sqrt{1-\omega^2\Delta t^2}}{2\sqrt{1-\omega^2\Delta t^2}}\left(y_0\right), \qquad b = \frac{-1+\sqrt{1-\omega^2\Delta t^2}}{2\sqrt{1-\omega^2\Delta t^2}}\left(y_0\right)$$

$\boxed{y_n = ar_+^n \;+\; br_-^n \quad \text{As } \omega\Delta t \to 0, \text{ then } a \to y_0 \text{ and } b \to 0}$ $\quad$ ~ 65-70 min

• 1st term: Physical mode: as $\omega\Delta t \;\to\; 0$, $\quad ar_+^n \to$ exact solution

• 2nd term: Computational mode: as $\omega\Delta t \;\to\; 0$, $br_-^n \to 0$

– This comp mode is the price we pay for 2nd order accuracy

– Essentialy all is OK, w/ higher resolution, $br_-^n \to 0$

HW3: write it in polar cooordinate {~ 66 min}

– calculate a & b; then α. then plot the 2 modes sep & together

– *set* $\omega = 1$

draw picture of the vectors: α is..... fill in

Alternate form of solution (writing in polar coordinate form):

$\boxed{y_n = ae^{in\alpha} + b(-1)^n e^{-in\alpha}}$, where $\alpha = \sin^{-1}(\omega\Delta t)$

Case 3: $\omega\Delta t = 1$

• Do it in HW. we will see: {~74 min. fig 1}

– computational mode will get very small

Case 4 {? see em's notes. JN fig 2}

HW: frequency (~75 min)

SUMMARY :

• $\omega\Delta t < 1 \Rightarrow$ Conditionally stable

• Existance of compuational mode

• This puts constraint on how big $\omega\Delta t$ can be: ie can't be bigger than 1

CHALLENGES :

• find ways to control comp mode

• in non-linear, comp mode becomes mildly unstable

Mon Feb 12

Homework notes

- In a mathematical proof, start with what is given, not with the concluson.
- QED: Latin, as it was shown/supposed to be shown ?

Aliasing: undersampling.

$\omega\Delta t = 2\pi\left(\frac{\Delta t}{p}\right)$ meadure of how well the osc is resolved. The smallest

for a given period, we do not want Δt to be too big. $\Delta t \le \frac{p}{2}$ or, $\omega\Delta t \le \pi$

Do not pick Δt to be too big! Make sure we have sampled osc sufficiently.

Aliasing examples: 1) wagon wheel effect: appears as if the wheel has stopped or is going backwards.

Review : Time differencng schemes.

Two - level schemes : $(n, n+1)$, all are 1st order accurate)

1) Euler (E) (Explicit): $y_{n+1} = y_n + \Delta t f(y_n)$

2) Backward (B) (Implicit): $y_{n+1} = y_n + \Delta t f(y_{n+1})$

   called B b/c $\rightarrow$ $\left[y_n = y_{n+1} + (-\Delta t) + f(y_{n+1})\right]$

3) Trapezoidal (T) (Implicit): $y_{n+1} = y_n + \frac{\Delta t}{2}\left[f(y_n) + f(y_{n+1})\right]$

Three - level schemes : $(n-1, n, n+1)$,

4) Leapfrog (LF): $y_{n+1} = y_{n-1} + 2\Delta t f(y_n)$

- has a computational mode. No one goes beyond 3 level. would get more comp. modes

– – – – – – – – – – – – – –

We have applied these methods to the oscillation eqn. Have found these properties:

E: unstable for all values of $\omega\Delta t$. $\therefore$

B: stable for all $\omega\Delta t$.,

   Cons: • has artificial dissipation, damps towards 0.

   $\rightarrow 0$ as $\Delta t \rightarrow \infty$

T: • stable for all $\omega\Delta t$.

   • no artificial dissipation

   • overestimates period

L: • Pros: i) 2nd order accurate; ii) stable; iii) explicit

   • stable if $\omega\Delta t < 1$ (conditionally stable)

   • Cons: i) puts restriction on Δt ii) computational mode

Multi - stage schemes (not multi step) from Duran

• AKA: prediictor-corrector schemes

• trying to get nice props of implicit, but w/outimplicit

Examples:

1) **Euler - Backward** = Forward-Backward = Matsuno (Alt names)

• 1st come up w/ 1st guess, then try use guess to get improved.

• Explicit method

• 2 time level

• Conditionally stable:

• 1st order accurate

• has artificial dissipation: sometimes it is a goood thing.

• can be used to start LF

$y_* = y_n + \Delta t \ f(y_n)$; stage 1

$y_{n+1} = y_n + \Delta t \ f(y_*)$; idea = $y_* \cong y_{n+1}$;

2) **Runge - Kutta schemes** (class of multi-stage schemes)

• they have 1st, 2nd, 3rd order schemes

• Cons: all are unstable when appliied to osc eqn, however Duuran advocates using them occasionally

----

Controlling the Computational Mode (for non-linear eqns)

• it comes since we are using 3 level scheme

• has a tendency to grow, thus there are methods to try to control it. 2 approaches:

1) periodically restart using a 2 level scheme

See fig 1

LF 17-19

LF 18-20

Restart

discard 19 (or avg 19 & 20)

20 → 21 using 2-level meth

resume LF....

∴ can use:

Euler backward: Con: degrades accuracy

Runge-Kutta: can be used carefully (Duran)

-programming: if n is divisible by 20, then employ 1 of above methods.

-these are empirical programming techniques.

2) modify differencing scheme: very popular

2. a) Time-filters

i) Asselin filter: tends to suppress comp mode.

note: comp mode in fig 2.

ii) Robert filter: Con: degrades accuracy; turns it from 2nd order to 1st order accurate

Pro: very popular, though Duran says it is dubious.

2. b) Leap-Frog Trapezoidal scheme: advocated by Duran

- It is a predictor-corrector
- 2nd order accuate

$$y_* = y_{n-1} + 2\Delta t\ f\left(y_n\right)$$

$$y_{n+1} = y_n + \frac{\Delta t}{2}\left[f\left(y_n\right) + f\left(y_*\right)\right]$$

FEB 14

Duran, table 2.1

solution $\sim \lambda^n e^{i(\ )}$

$|\lambda| > 1 \Rightarrow$ exponential growth (unstable)

$|\lambda| = 1 \Rightarrow$ constant amplitude

$|\lambda| < 1 \Rightarrow$ solution is damped (artificial dissipation)

"Phase Error" $= \dfrac{\omega_c}{\omega}$, where ω_c = computed frequency

The table is generated from from the oscillation eqn.

If a method is superior for the oscillatin eqn, then it is superior for the full set of atm eqns.

A PDE

• linear (simple); only 1 spatial dimension,

$\boxed{\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0}$ Advection Eqn; c = constant; adv in x direction w/ constant speed

compare: $\dfrac{\partial u}{\partial t} + V \bullet \nabla u$

$$\equiv \frac{du}{dt} \ \{\text{toal deriv}\}$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \\ &u(x,0) = f(x) \end{aligned}\right\} IVP, \text{ has a uniique soln: } u(x,t) = f(x - ct)$$

Can be easily proved by pluggin in:

$$\frac{\partial u}{\partial x} = f'(x-ct)\frac{\partial(x-ct)}{\partial x} = f'(x-ct)$$

$$\frac{\partial u}{\partial t} = f'(x-ct)\frac{\partial(x-ct)}{\partial t} = -cf'(x-ct)$$

Now do specific example:

$$u(x,0) = Ae^{-\frac{x^2}{L^2}}$$

$$u(x,t) = Ae^{-\frac{(x-ct)^2}{L^2}}$$

See fig 1 & 2

Note: distrubance is moving at speed c, w/out change of shape

i.e if we put a pssive tracer in a stream, and ignore turbulence.

Thus, w/ any diif scheme we want:

• c to stay the same

• no sign change

• just whole wave to move, same shape etc.

Special case: This describes advection, hoping to describe something in real life w/ a simple eqn.

Now we are dealing with space differencing. we used LF for time diiff before

$u(x,0) = Ae^{ikx}$

$u(x,t) = Ae^{ik(x-ct)}$

Fig 3: sinusoidal wave: • can only move in 1 direction

$c =$ phase speed, $k = \dfrac{2\pi}{\text{wavelength}}$ wave number $\left\{\begin{array}{l}\text{k serves same purpose in space} \\ \text{as } \omega \text{ does in time}\end{array}\right\}$

Goal: Calc finite diff:

• will get errors in: phase speed, ampltude

SPATIAL DIFFERENCING (FIG 4)

$$\left(\frac{\partial U}{\partial x}\right)_{x_0,t} \cong \frac{u(x_0+\Delta x,t)-u(x_0-\Delta x,t)}{2\Delta x};$$

2nd order acurate in space
centered diff; RHS replaces dervi of exact soln

We will deriive this in a special way & end up w/ osc eqn, and can use past conclusions

We will end up w/ CFL criteria

Feb19

LF, $\omega\Delta t = 0.999$

when they are in phase=> Constuctive interference

when 1/2 cycle out of phase $\Rightarrow$ destructive interference

LF, $\omega\Delta$ close to 0, comp mode goes away

SPATIAL DIFF

much of what was true for time diff is same for space diff

Fig 1

$$\boxed{\frac{\partial u}{\partial x} \cong \frac{u(x+\Delta x,t)-u(x-\Delta x,t)}{2\Delta x} \equiv \delta_{2x} u}$$ δ_{2x} is a differencing operating, approx the deriv

Example: $\delta_{2x}\left(e^{ikx}\right) =$ $\boxed{\text{Recall: } \frac{\partial}{\partial \text{x}}\left(e^{ikx}\right) = ike^{ikx}}$

$$= \left(\frac{e^{ikx+\Delta x} - e^{-ikx+\Delta x}}{2\Delta x}\right)e^{ikx} = \left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}\right)e^{ikx} = i\left(\frac{\sin k\Delta x}{\Delta x}\right)e^{ikx}$$

Aliasing: same applies here. Shortest resolvable $\boxed{\lambda = 2\Delta x,}$

$\therefore \boxed{(k\Delta x)_{\max} = \pi}$

$2\Delta x$ can be one of the most troublesome.

Approx val=$\dfrac{\sin k\Delta x}{\Delta x}$

Ratio $\dfrac{\text{approx}}{\text{exact}} = \dfrac{\frac{\sin k\Delta x}{\Delta x}}{k} \Rightarrow \dfrac{\sin k\Delta x}{k\Delta x}$

Fig 2.

The deriv of $2\Delta x$ wave is 0!! Prettty bad

Look at $2\Delta x wave$ (Fig 3)

Foregone conclusion if we use centered diff on $2\Delta x$ wave then we get 0

Centered spatioal diff never gives god results for $2\Delta x wave$, when we have sharp gariadients this can cause probs.

24 min, fill in: Crux of the problem....

Back to the advection eqn (linear, 1-dimensional)

$$\left.\begin{array}{l}\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0,\ \text{moves it speed c}\\ u(x,0)=Ae^{ikx}\end{array}\right\}\ (1)$$

, $\Rightarrow$ Non-dispersive Traveling wave,

$\lambda=\frac{2\pi}{k}, c=$ phase speed

Exact: $u(x,t)=A_0e^{ik(x-ct)}$

can be rewritten as: $u(x,t)=\left(A_0e^{-ikct}\right)e^{ikx}$

Finite diff in space, ...

32 min

Now:

Replace (1) by $\frac{\partial u}{\partial t}+c\delta_{2x}u\delta_{2x}=0 \quad (2)$

• Sincewe rewrote eqn above in new fokΔxrm, fiind sol to match. fill in

• Assume a soln of the form $u(x,t)=A(t)e^{ikx} \quad (3)$

• Subst (3) into $(2) \Rightarrow \frac{dA}{dt}e^{ikx}+cA(t)\delta_{2x}\left(e^{ikx}\right)=0$ | This not total deriv, it is ordinary deriv

$$\frac{dA}{dt}e^{ikx}+ic\left(\frac{\sin k\Delta x}{\Delta x}\right)e^{ikx}A(t)=0 \Rightarrow \frac{dA}{dt}\cancel{e^{ikx}}+ic\left(\frac{\sin k\Delta x}{\Delta x}\right)\cancel{e^{ikx}}A(t)=0$$

now we have ordinary diff eqn for A. Rewrite:

$$\left.\begin{array}{l}\frac{dA}{dt}=i\left(-c\frac{\sin k\Delta x}{\Delta x}\right)A\\ A(0)=A_0\end{array}\right\}\ (4)$$

| This is osc eqn with $\omega=-c\frac{\sin k\Delta x}{\Delta x}$ where ω is a constant

$\therefore$ we can say previous results apply

$\therefore$ Soln to $(4)=A(t)=A_0e^{i\omega t} \Rightarrow u(x,t)=A_0e^{i\omega t}e^{ikx} \quad (5)$

$\Rightarrow u(x,t)=A_0e^{i((kx+\omega t))} \Rightarrow \boxed{u(x,t)=A_0e^{ik\left(x+\frac{\omega}{k}t\right)}}$ | aprox soln , space deriv 48 min

$\Rightarrow A_0\exp\left[ik\left(x-c\frac{\sin k\Delta x}{\Delta x}t\right)\right]$ | notice this is similar to exact sol except we have c' term

$\Rightarrow A_0e^{ik(x-c't)}$ | where $c'=c\frac{\sin k\Delta x}{\Delta x}$ travelling wave with phase speed c'

Amp=A_0, Travellingwave, $\lambda=\frac{2\pi}{k}$, phase speed=$c\frac{\sin k\Delta x}{\Delta x}$

Pnly doff betwn approx &exact is phase speed, this is where error shows up

c' depends on k

Note: $\{\text{Fig } 5, \sim 57 \text{ min}\}$

• $2\Delta x$ is stationary, bad error. It should be moving along at c

• For well resolved waves, $c' \approx c$

• Approx soln is a dispersive wave

• In the real atm, there is not 1 wave but many, have to do fourier analysis
______fill in

WAVE DISPERSION

$u(x,0) = a_1 e^{ik_1 x} + a_2 e^{ik_2 x}$, simple fourier decompsition

Exact soln: $u(x,t) = a_1 e^{ik_1(x-ct)} + a_2 e^{ik_2(x-ct)}$

Patterns of constuc/destruc interfernc will be same, since we just translated it.\

$\therefore$ non-dispersive, initial shape dos not change

x: sound, light in vacuum, shallow water

But, in the case where $c = c(k) \Rightarrow$ Dispersive

$c_1 \equiv c(k_1) \neq c(k_2) \equiv c_2$

$u(x,t) = a_1 e^{ik_1(x-c_1 t)} + a_2 e^{ik_2(x-c_2 t)}$

Matlab demo:

since crest & troughs are at diif sppeds, shap chnages with time

Ex: deep water waves, (longer λ goes faster than shorter)

We do not diispersion created by the diff. scheme.

it is worst for the short λ.

if we have well resolved waves, we get good soln.

With short: unravels, (Fig. 6) $\{76 \text{ min}\}$

• this is prototype ex: fn diff not good w/ sharp gradients.
ex: we can start with weak grads, but then they grow to strong ones

----

Summarize:

• spat diff makes phase speed a func of k, then you get numerical dispersion
phase speed depends on wavenumber

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next time:

Group velocity: like energy propagation speed

Feb 21

HW Comments: Fig. not Figure in text (exception is at beg. of sentence)

Last time

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad (1)$$

$$u(x,0) = A_0 e^{ikx}$$

$$(1) \rightarrow \frac{\partial u}{\partial t} + c\delta_{2x} = 0$$

form: $u(x,t) = A(t)e^{ikx} \;\Rightarrow\; \frac{dA}{dt} = i\omega A \;\; (2)$

this leads to an ODE, happens to be the Osc eqn with a particular form of ω

$$\omega = -c\frac{\sin k\Delta x}{\Delta x} \;\; (3)$$

Note: exact is non-dispersive. numerical has dispersive

fill in ___ we would n't want to use this to comper with ____ (12 min)

Time diff of (2); just looking at time dimension now

Methods of time diff (so far)

Method	Comments
1 Euler	• unstable
2 Backward (Implicit)	• stable (for all Δt)
	• numerical dissipation, i.e amplitude decays w/ time
3 Trapezoidal (Implict)	• Absolutely stable; amplitude is correct.
	• no dissipation
4 LF (Explicit)	• Conditionally stable if $\lvert\omega\rvert\Delta t < 1$
	• Computational mode is a con
	• Physical mode is ________

Notes on terrminology:

• If numerical scheme introduces an unboundedness, then it is "unstable"

i.e. if it grows w/out bound. we do not want to create spurious growth.

• method is stable/unstable

• solution is either bounded or unbounded.

~22 min

29 min

Stability Condition for LF

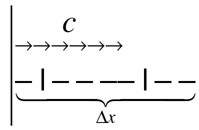
$\omega = -c\dfrac{\sin k\Delta x}{\Delta x}$

$\left(\dfrac{|c||\sin k\Delta x|}{\Delta x}\right)\Delta t < 1 \quad \Rightarrow \quad \dfrac{|c|\Delta t}{\Delta x}|\sin k\Delta x| < 1$

- We want to require that we have stability for all k (wave numbers)
- $|\sin|$ is always $\leq 1 \quad \therefore \quad \boxed{\dfrac{|c|\Delta t}{\Delta x} < 1}$ | Courant-Friedriichs Levy oncondition CFL
- CFL<1 is the condition for 1 ________ dimensional cases

  in other sit there is some other condition that limits the ratio $\dfrac{\Delta x}{|c|}$
- CFL $\Rightarrow \Delta t < \dfrac{\Delta x}{|c|}$ | this condition is exact for this case w/ advection eqn

$\dfrac{\Delta x}{|c|} =$ the time for wave to move 1 grid interval



the time step has to be < the time it takes for the wave to move 1 grid interval

- This creates problems when there are many difff waves w/ diff speeds

Global Atm Model

- Diff types of waves. $\Delta x = 100$ km
- $\therefore$ CFL $\Rightarrow \Delta t < \dfrac{\Delta x}{|c_{\max}|}$, where $c_{\max}$ = max wave speed
- Sound waves are the fastest waves in the fluid of the atmosphere

$c_s \cong 350 \text{ m s}^{-1} \Rightarrow \Delta t < \dfrac{10^5 \text{ m}}{350 \text{ m s}^{-1}} \cong 300 \text{ s} = 5 \text{ min}$

- This is excessively small, more resolution than required for meteorological waves
  this is ~ 6 times to small (ie 30 min)
- if we cut Δt in half, we have to cut Δx in half (check on...)

Ways to relax this conditionn b/c it is so stringent. here, cut out sound waves:

1) Hydrostatic approximation:
   - it filters out all vertically propagating sound waves.
   - there are however, still some horizontal sound waves that are hydrostatic
     ie horiz propagating waves that move at speed at sound.
     they come with volcanoes, etc...

2) Semi-implicit method allows longer time steps. This slows down the speed of the fast moving, non-meteorological waves, diistorts the picture some, but they are not important. Cons: more difficult to work with.

COMBINE TIME AND SPACE DIFFERENCING   Ex: $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$

Notation: U_j^n, where $n =$ time index, and $j =$ space index

Goal: appproximate time deriv

$$\left(\frac{\partial u}{\partial t}\right)_j^n \cong \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}$$

$$\left(\frac{\partial u}{\partial x}\right)_j^n \cong \frac{u_{j-1}^n - u_{j-1}^n}{2\Delta x} \Rightarrow \boxed{u_j^{n+1} = u_j^{n-1} - \frac{c\Delta t}{\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)}$$

• Have to use diff method to get $n_{\cdot\cdot 1}$

$n = 0 \to n = 1$

$$u_j^1 = u_j^0 - \frac{c\Delta t}{2\Delta x}\left(u_{j+1}^0 - u_{j-1}^0\right)$$

• Problem: x domain has to be restricted; need boundaries

j || 0 1 2 3 4... ...$j_{\max}$ | $j_{\max+1}$

---|---|---|---|---------------------|------*

x || 0 L

these lateral boundaries are purely computational exception : full latitude circle

Simplest boundary conditions

• cyclic BC, ie. $u(0,t) = u(L,t)$ (works well with periodic functions)

$\therefore$ above cyclic case would be $\Delta x = \dfrac{L}{j_{\max}}$ $u_0^n = u_{j_{\max}}^n$

want Δx to be considerably less than L

• Real problems come at end points; only need to calc one of them (~75 min)

• $u_{j+1}^n - u_{j-1}^n$

• $j = 1$: $u_2^n - u_0^n = u_2^n - u_{j\max}^n$

• $j = j_{\max}$ $u_{j\max+1}^n - u_{j\max-1}^n$

• $u_1^n - u_{j\max-1}^n$

• problem need value at 0

• Look at general formula for everyhting but endpoints

• special cases at end points: use diff formulae

• Initial conditions need to be periodic

• for longer waves, w/ good res, we will get good results.

• med res will give ok results

• _____ will be stationary

• do not go out many time steps, just enough to see if the wave ismoving

• easiast way is to have an array with 2 subscripts.

when debuggin it, want to see if soln is in the right ballark.

• Exact: thing should move at speed c

Feb 26, HW5, 0 min...

$u = leapFrog(u0, u1, s, nMax)$ title: LF solution of a well resolved wave.

$$u(j,n)\begin{cases}\text{for } n = 1 : nMax \\ \quad \begin{cases}\text{for j} = 1 : jMax \\ \\ end\end{cases} \\ end\end{cases}$$

$u(j,n) =$

• See that sin wave is propagating at constant speed; $t = 0, \Delta t, 2\Delta t, 3\Delta t...$

• plot a bunch w/ diff colors

Exam 1 review (~ min)

• no heavy calculations

• short deriivations

• no programming

• HW is designed to illustrate the theory, imp. to know theoretical stuff for exam

• osc eqn,

• 4 differencing schemes

• be able to fill in Durans table: stable; damped, ...

• CFL

================

EFFECT OF PHASE SPEED

1) Spatial differencing (discretization)

$\Rightarrow$ causes waves to be dispersive, phase speed depends on wave #: $c' = c'(k)$

Fig 1.

as we go to the left, get bettter spatial resolution of the wave

as $k\Delta x \to 0$, $\lambda \to \infty\infty$

only takes into account spatial diff, not time diff.

- if diff time scheme, then it would alter this

How to improve? this curve is 2nd order diff,

��What would happen w/ 4th order scheme?

$$\left(\frac{\partial u}{\partial x}\right)_j^n \cong \frac{4}{3}\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) - \frac{1}{3}\left(\frac{u_{j+2}^n - u_{j-2}^n}{2\Delta x}\right)$$

Trunc err: $\mathrm{O}(\Delta x^4)$

HOw to create? start w/ arbitrary linear combo, calc Trunc error, try to zero everything but 1st 4

each coef of Taylor has; all squared, cubed terms drop out.

Idea: since spatial difff is causing propblem, perhaps higher order willhelp

$\Rightarrow$ *Still* have problem that $2\Delta x$ wave is stationary, no improvement there. Reason:

whenever you have centered diff you get zero.

• there will be improvement if we have smooth waves, (26 min)

Case : Advection

• what happens if func has extreme gradients?, like a spike eg. Fig 2 & 3

• When might that occur? say frontal zone,

See Steff. fig 2

• this phenomenon (trailing waves) can be explained by group velocity (energy propagation).

this si not a computational mode, you would get this w/ any scheme. it is purely spatioal, ind of time diff scheme.

Group velocity, c_g, the speed that energy propagates; energy prop velocityy

In 1D: $c_g \equiv \frac{\partial \omega}{\partial k}$ where ω=kc In 3D, it wold be gradiient, but we won't address that now

$c_g = \frac{\partial}{\partial k}(kc) = c + k\frac{\partial c}{\partial k}$ | For dispersive waves $c_g \neq c$ could be greater or less, depending on...

$c_g = c$ For non-dispersive waves;

eg: deep water waves (dispersive), $cg = 1/2c$, *energy* goes slower, wave crests appear, move through, and disapear.

eg: shallow water waves are non-dispersive

eg: tsunami are non-dispersive, b/c of wavelength

eg. internal gravity waves, 2D, energy goes at cross angle (?)

Apply this concept to the discrete solutions of advection eqn.

$\frac{\partial c'}{\partial k} < 0$, *therefore* $\Rightarrow$ $c_g < c$; we already know c est is smaller than exact.

we could get a sign change, then they could go backward. (?)

Fig 4

• note it changes sign;

• this explains this phenomenon.

- once LF code is writen, it is easy to apply it to spike.

• you end up w/ a wave train, ...

• it's as if the disturbance is ratiang energy backward,

in a front,

Note 1B, if we use 4th order, it improves at first, but near $2\Delta x$, the neg slope is even more neg, thus the

$\therefore$ thus no method that ever uses centered diff will improve thisproblem higher the order, steper the slope, faster waves will prop upsteam

This is only a proble w/ steep gradients, if we did not have

Next time: upstream diff (1st order accurate). This is the springboard to semi-lagrangian etc..

it gets rid of those waves, but produces an effect which is almost the oppposiste of

• Cenetered always has even number trunc erros

• 1st order have odd

Feb 28
Hw 4 notes

Q1: recursion formula produces the numerical solution

Analytic solution = algebraic sol that solves the recursion formula
Usually you do not have analytic solution.
Ie. get the sol. From 2 diff ways, sols should be exact
$y_n = ar_+^n + b_-^n$, $a, b, r_\pm$ are functions of $\omega\Delta t$

Q2: if abs r is > 1, then sol grows

DISPERSION

fig. 1

- the greater the oder on centered diff, the worse the neg error grows,

Uncentered differencing

e.g. 'upstream' differencing {Fig 2} • downstream diff doesn't make sense

- assume $c > 0$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0 \quad \Rightarrow \quad u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

Conditionally stable: stable if $0 \le \frac{c\Delta t}{\Delta x} \le 1$

Conclusions: {See Steff. Fig 3; }

- no neg values!! ∴ it is possible to craft a solution that preserves the sign ***
  imp if we advecting density for example
- do not see waves downstream waves
- this behavior is very diff from before.
- the solution is exhibiting spatial dissipation; it is losing ground quickly

Now, We want a method that works for anything; See Duran fig. 2.13a

Dispersion ------------- / ________

Diissipation ....

b) Intermediate solution: improvement out to 5 grid pints

Almost same _._._._._._. exact / ________ 4th

--------------- 2nd order

upstream .... 1st order, extreme reduction in amplitude; wipes out everything

- 4th order ⇒ improvement over 2nd order
- 1st order:

Uncenterd: produces dissipation

Centered: all produce dispersion, no dissipation to

odd: have dissipation built in

Summary:

• people try to find balance between dissipation & dispersion

• no perfect method: impossible to advect $2\Delta x$ spike

Upstream differencing $u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$

$$u_j^{n+1} = u_j^n - \frac{c\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

$$u_j^{n+1} = \frac{c\Delta t}{\Delta x}u_{j-1}^n + \left(1 - \frac{c\Delta t}{\Delta x}\right)u_j^n$$

• this is a weighted avg. avg must be between 2 numbers $\{fig4\}$

• $\min\left\{u_{j-1}^n, u_j^n\right\} \le u_j^{n+1} \le \max\left\{u_{j-1}^n, u_j^n\right\}$

• what does this say about sign? suppose $u_k^n \ge 0$ for all k

⇒ then $u_k^{n+1} \ge 0$ for all k

• If we keep avging, then max point keeps coming down;

• if $\frac{c\Delta t}{\Delta x}$ becomes neg, then we would be extrapolating

• Upstream: not useful by itself, but useful for developing diff ideas

----

Atm: if we use centerd diff: easy to get wave dispersion

• sometimes we want to add in dissipation; then, use a scheme that has dissipation built in prob, we do not have control. so, you could add dissipation term, just to smooth out solution.

• this is all about compromise

Test:

• re. spike, show what happens if we have scheme with:
- dispersion
- dissipation

March 7 ARTIFICIAL DISSIPATION

Scale-dependent dissipation. want:

→ damp out short λ

→ small damping for longer λ

Goal: get amplitude ~ $e^{-d_k t}$ {Fig .1}

Have something the multiplies the certain $k\Delta x$ by a damping factor.

Methods for damping out shorter wavelengths:

• upstream diff, only stable if wind doesn't change for example. thus , not the best solution.

• Diffusion: add diffusion term;

Diffusion Equation: $\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial x^2} \quad (1-D)$,

$u(x,0) = A_0 e^{ikx}$

Look for solutions of the following form (has same spatial charaecterisitcs)

$u(x,t) = A(t)e^{ikx} \quad (1)$

$$\frac{dA}{dt}e^{ikx} = \upsilon A(t)\left[-k^2 e^{ikx}\right] \Rightarrow \frac{dA}{dt} = -\upsilon k^2 A \Rightarrow A(t) = A_0 e^{-\upsilon k^2 t}$$

Then substitute into $(1) :\Rightarrow \boxed{u(x,t) = A_0 e^{-\upsilon k^2 t} e^{ikx}}$

We get cos & sin waves, amplitude depends on time. $\{Fig2\}$

Preferentially damps out smaller scale

Example: $\{Fig3\}$

$u(x,0) = A_1 e^{ik_1 x} + A_2 e^{ik_2 x}$

$u(x,t) = A_1 e^{-\upsilon k_1^2 t} e^{ik_1 x} + A_2 e^{-\upsilon k_2^2 t} e^{ik_2 x}$

• $k2$ component is being damped much faster: $A_2 e^{-\upsilon k_2^2 t} e^{ik_2 x}$

Suppose: $k_2 = 10k$ & $A_2 = A_1$

ie. initial amplitudes are equal

Then take ratio of 2 amps

$$\left|\frac{A_2(t)}{A_1(t)}\right| = \left|\frac{A_2 e^{-100\upsilon k_1^2 t}}{A_1 e^{-\upsilon k_1^2 t}}\right| = e^{-99 k_1^2 t}$$

Bigger λ will get damped much more. Higher wavenumber component
is disapearing compared o other component

• Diffusion is a smoothing process. short λ creates unsmooth effect
smalll scale stuff is selectively damped. We are left w/just the long λ.
However long wavelengths get slightly damped, but leess so. If we go out
far enough,

We get smoother soln. Diff attacks high wave number selectively, (ie $2\Delta x$, $3\Delta x$)
which is what we want

Goal: Combine advection and difffussion

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=\upsilon\frac{\partial^2 u}{\partial x^2}$$

Diff is not real, we add it in for computatinal purposes. We are smoothing out soln. Need to take υ to be small. If it were too large, even long wave stuff would be quickly damped out. Sole purpose dampen high wave numbe stuff, leave in low wave number stuff. $\{\text{Fig } 4\}$

2 ways to do artificial dissipation:

1) Add diffusion term to centered diff scheme

2) upstream: automatically comes w/ diff scheme. only problem it is only stable if wind comes in 1 direction only (doesn't change)

4th derivate diffusive term: $\frac{\partial u}{\partial t}=-\gamma\frac{\partial^4 u}{\partial x^4}$

$u(x,0)=A_0e^{ikx}$

$\Rightarrow u(x,t)=A_0e^{-\gamma k^4 t}e^{ikx}$

• the k^4 term drops off much faster than k^2 term, $\therefore$ better

k^4 takes out smaller chunk of long wavelength stuff.

Idea: selectivel attack high wavenumber, leave low wavenumber stuff alone

Antidote to disspersion.

March 12 Centered Differencing for Advection

$\rightarrow$ dispersion

$\rightarrow$ short-wavelength "noise"

• To damp noise, add a diffusive term

$\frac{\partial u}{\partial t}$ = advection term + difffusion term

• Diffusion $\rightarrow$ scale-selective dissipation (high wavenumbers damped most)

• Diffusion term acts as a filter: filters out noise(w/out affecting longer λ)

• This is more elegant, less brutal (use FFT to remove everything from that end of spectrum)

Diffusion Terms :

$\approx\frac{\partial^{2\ell}u}{\partial x^{2\ell}},\ \ell=1,2,3,...$

• more difficult to program as l gets bigger; 6 is a popular one

• if l=2, call it a 4th derivative filter. l=3, call it a 6th derivative filter.

Finite-differencing: Systematic aproach

$$(\delta_x u)_j\equiv\frac{u_{j+1/2}-u_{j11/2}}{\Delta x}\cong\left(\frac{\partial u}{\partial x}\right)_j\rightarrow(\delta_x^2 u)_j\cong\left(\frac{\partial^2 u}{\partial x^2}\right)_j$$

$$\boxed{\left(\delta_x u\right)_j \equiv \frac{u_{j+1/2} - u_{j11/2}}{\Delta x} \cong \left(\frac{\partial u}{\partial x}\right)_j} \rightarrow \boxed{\left(\delta_x^2 u\right)_j \cong \left(\frac{\partial^2 u}{\partial x^2}\right)_j}$$

$\delta_x^2 u_j$

$$= \delta_x\left(\delta_x u_j\right) \Rightarrow \delta_x\left(\frac{u_{j+1/2} - u_{j11/2}}{\Delta x}\right) \Rightarrow \frac{1}{\Delta x}\left(\delta_x u_{j+1/2} - \delta_x u_{j11/2}\right)$$

| do whatever operator says

$$= \frac{1}{\Delta x}\left(\frac{u_{j+1} - \delta_x u_j}{\Delta x} - \frac{u_j - u_{j-1}}{\Delta x}\right) \Rightarrow \boxed{\delta_x^2 u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta^2 x}}$$

- most people use 4th & 6th, (not the 2nd as often)

Durran

$u_j = A(t)e^{ikx}$ |Subst into difference eqn

$$\rightarrow \frac{dA}{dt} = -\gamma\left(\text{non-neg. expression in } k\Delta x\right)A$$

| $-\gamma\left(\text{non-neg. expression in } k\Delta x\right) =$ damping rate

- Adding diffusive term to do damping,
- γ is an adjustable constant
- Durran graphs coeff, can normalize it.
- Fig 1: Compare filters
  2nd order has too much damoing in long λ section
  4th is an improvement
  6th is even better (very little damping for long range, then
  strongly damp waves w/ λ between $4\Delta x$ &$2\Delta x$
  – more problematice at boundaries; have 6 exceptional points
- perfect filter would be nodamping, then sharp damping. It drops off the table
- we want to not lose ampl in long λ component.
- adding diff filter to remove high ____noise is very common.
- Aselin filter does the same thing but in time. (LF switches sign)

Mote Book: chapt. by Williamson; JS does not like it

SHALLOW WATER EQNS TWO VARIABLES

Useful in Atm science b/c there are these terms:

- Advection
- PGF
- divergence

Assumptions :

- depth is small compared to ; vert scale $\ll$ horizontal
  Shallow water: elipses are very flat, just line, vs.
  Deep water waves: particles travel in circles/full elipses
- $\therefore$ ignore vert accelerations;
- PG ~ $\frac{\partial h}{\partial x}$; pressure is proportianal to depth above it.
- PGF $= -g\frac{\partial h}{\partial x}$

Advantages :

- Hydrostatic assumption
- Simplest set of eqn that allow one to model PGF & divergence
- Can be solved analytically.
- waves can propagate in 2 directions; can even model reflections, standing waves, intereference.

Equations :

Assumptions:

- 1-D
- no rotation
- linearized.
- h,&u have to remain small

Variables: u, h These are perturbation values

(1) $\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$	$-g\frac{\partial h}{\partial x} = PGF$ term; U is constant
(2) $\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = -H\frac{\partial u}{\partial x}$	$-H\frac{\partial u}{\partial x} =$ Divergence term

Finite differencing of shallow - water eqns

Method 1

- LF time diff, 2nd order centered in space

$$(1)\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + U\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = -g\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x}$$

$$(2)\frac{h_j^{n+1} - h_j^{n-1}}{2\Delta t} + U\frac{h_{j+1}^n - h_{j-1}^n}{2\Delta x} = -H\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

Fig 3: staggered grid gives you more accurate phase

Goals:

- get best representation for phase speed
  - $\rightarrow$ staggered grid gives best representation

March 14: Notation

$$\delta_{mx}u_j^n \equiv \frac{u^n_{j+m/2} - u^n_{j-m/2}}{m\Delta x}$$

$$\delta_{pt}u_j^n \equiv \frac{u_j^{n+p/2} - u_j^{n-p/2}}{p\Delta t}$$

Note : reg forward diff could be written in this shorthand: $\delta_t u_j^{n+1/2}$

eg.: Advection Equation

- Leapfrog + 2nd order centered spatial diff

$$\delta_{2t}u_j^n + c\delta_{2x}u_j^n = 0$$

- Leapfrog + 4nd order centered spatial diff

$$\delta_{2t}u_j^n + c\left(\frac{4}{3}\delta_{2x}u_j^n - \frac{1}{3}\delta_{4x}u_j^n\right) = 0$$

Shallow Water eqns

- looking for wave-like solutions

every travelling wave looks like:

Variables: u, h These are perturbation values

(1) $\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}$	$-g\frac{\partial h}{\partial x} = PGF$ term; U is constant
(2) $\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = -H\frac{\partial u}{\partial x}$	$-H\frac{\partial u}{\partial x} =$ Divergence term

$$\left.\begin{array}{l} (1)\ \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} \\ (2)\ \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = -H\frac{\partial u}{\partial x} \rightarrow \begin{cases} u = ae^{ik(x-ct)} \\ h = be^{ik(x-ct)} \end{cases} \end{array}\right\} \rightarrow c = U \pm \sqrt{gH}$$

|Spceial case

H should be small compared to λ

1st RHS term ___________

2nd RHS term -> divergence

waves come from div & conv,

you can tell which term gives us waves,

they are markers {26 min. fill in}

Finite diff example: Leapfrog+2nd order space fillin & chaeck SW prime eqhs

$(1')\delta_{2t}u_j^n + U\delta_{2x}u_j^n = -g\delta_{2x}h_j^n$

$(2'\ )$ ______________________

To do stability analysis ()

Look for wave soln, want to get phase speed

If complex c_..=> growing soln; unstable (2 roots)

$\mathrm{Re}\,sults:\{36\min, fig1\}$

instead of c we have

$$|U| + \sqrt{gH}\,\frac{\Delta t}{\Delta x} < 1$$

$meaning: \Delta x speed rel to fluid + vel of fluid:$

$40 + 10 = 50 m/s$

$but if upsteeam, then subtract$

$30 - 10 = 20$

• this give us a restriction. we can rewrite it

$$|U| + \sqrt{gH}\,\Delta t < \Delta x$$

ie. dist can't go farther than 1 gris step, else unstable

STAGGERED GRID $\{$Fig 2$\}$

calculate h (p in the atm)

• how to calculate $(\mathrm{PGF})_j^n = -g\,\dfrac{h_{j+1/2}^n - h_{j-1/2}^n}{\Delta x} = -g\delta_x h_j^n$

$$\begin{pmatrix} Div \\ term \end{pmatrix}_{j+1/2}^n = -H\,\frac{u_{j+1}^n - u_j^n}{\Delta x} = -H\delta_x U_{j+1/2}^n$$

Compare Trunc errors: unstaggered vs stagggered

• Staggered should be more accurate

1) Unstaggered. evaluate deriv at (fig 3)

$$\left(\frac{\partial h}{\partial x}\right)$$

$$Tr = \frac{h_{j+1/2}^n - h_{j-1/2}^n}{2\Delta x} - \left(\frac{\partial h}{\partial x}\right)_j$$

$$= \frac{1}{6}\frac{\partial^3 h}{\partial x^3}\Delta x^2 + \ldots. \qquad \bullet\ 2nd order$$

2)*Staggered*

$\left(\frac{\partial h}{\partial x}\right)$

$$Tr=\frac{h^n_{j+1/2}-h^n_{j-1/2}}{\Delta x}-\left(\frac{\partial h}{\partial x}\right)=\frac{1}{6}\frac{\partial^3 h}{\partial x^3}\left(\frac{\Delta x}{2}\right)^2+....$$

$$Tr=\frac{1}{24}\frac{\partial^3 h}{\partial x^3}\Delta x^2$$

- $Tr(\text{stag})=\frac{1}{4}*Tr(unstaggered)$
- b/c we are closer to the lim it,
- almost like we got something for nothing
- sameexceptreplaceinΔx with
- we get same result for the divergence term: improved accuracy
- if this is what we want,if we study waves,
  want the best rep of ______speed as possible
- fill in ______(56 min)

Limitations?

Unstaggered vs. staggered stability

$$U_n:|U|+\sqrt{gH}\frac{\Delta t}{\Delta x}<1$$

$$St:|U|+\sqrt{gH}\frac{\Delta t}{\Delta x}<\frac{1}{2}$$

- ie... for a given del x, del t is 1/2 of what is was for the unstag grid
  t will take twice as many time steps to get to the same place

that is the proce we pay

but accuracy is up by 4, other is by 2,

so we are still l ahead of the game.

Question: how imp ois it to get most accurate phase speed possible

Phase speeds {*fig*4}

- Computed c are dispersive,

$$\frac{c_{un}}{c}=\frac{\sin k\Delta x}{k\Delta x},\qquad \frac{c_{st}}{c}=\frac{\sin\left(\frac{k\Delta x}{2}\right)}{\left(\frac{k\Delta x}{2}\right)}\qquad \text{Recall: } 0<k\Delta x\le\pi$$

Group velocity of $2\Delta x$ wave

Summary • Staggered: improvement the calc/sim of waves

- price: time steps are limited. but if we want accuracy, we have to pay some price.

$\to c_g$ of $2\Delta x$ wave (worst one)

- $c_{g,u}=-c$ (200% error)
- $c_{g,st}=0$ (100% error)

Staggering: get improvement

- with many variables it gets more complicated
- very popular
- the c grid gives the best gravity waves ?

March 19

Fill in 1st 10 min from Emerson.

HW7: get notes from Emerson

Time-diff	Advection	Diffusion	
Forward	U	S	
Leapfrog	S	U	

U=Unstable
S= conditionally Stable)

{fig 1)

get note 2 fromn EL?

LF with lagged diffusion (evaluate it at time $n-1$)

• Conditionally stable

• diff term effcts stab condition: maes it stricter, reduces max Δt

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \upsilon \frac{\partial^2 u}{\partial x^2}$$

$$\boxed{\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = \upsilon \delta_x^2 u_j^{n-1}}$$

above is not rel to Q3

Hw 7, Q3 (~ 13 min)

• when c=0, just do straight forward diff.

$$\boxed{\frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} = -\gamma_4 \delta_x^4 u_j^n}$$

•if we put in any advection, it will blow up.

• Alt: option: use implicit for time differencing. We will not do it now. Prob: it leads to a set of linear eqns

SPECTRAL MODELS

• Steeper learning curve: need to know about fourieer series etc.

•

March 21

{*fig*.1}

fill in from emerson

• Vectors

• Basis

• Dot Product

• Orthogonal

$\therefore$ V_j can be written as dot products

$$v_j = \frac{v \cdot \vec{b}_j}{\vec{b}_j \cdot \vec{b}_j} = \frac{v \cdot \vec{b}_j}{\left\|\vec{b}_j\right\|^2}$$

• you get projection on basis vector.

• only numerator couns

• a vector is 0 if and only of each component is zero.

$\boxed{\vec{V} = 0 \text{ iff } \vec{V} \cdot \vec{b}_j = 0}$ all for j iff dot product of any basis is zero

Complex vectors

$$x \cdot y = \sum_{i=1}^{n} x_i y_i^*$$

$$\left\|\vec{x}\right\|^2 = \vec{x} \cdot \vec{x} = \sum_{i=1}^{n} x_i x_i^* = \sum_{i=1}^{n} \left|x_i\right|^2 > 0, \text{ where } * = \text{complex conjugate}$$

•*we* want to get a positive number

• evertything is same in complex space, except need to use the complex conjugate.
def of dot product is slightly different.

Fill in .....

$\boxed{\text{Function Spaces}}$ = set of functions (s) satisfying

1) If $f \in S$, then $\alpha f \in S$ (α is a scalar)

•*ie* any multiple must be in set

2) If $f_1 \in S$ & $f_2 \in S$ then $f_1 + f_2 \in S$

• we can crearte linear combinations

• *ex* : if we had set where all values were defined by bound,

ie. we can not have bounded sets

Example :

Example:

$S_{2\pi} = \{ f : f(x+2\pi) = f(x) \text{ all } x \}$

Basis of $S_{2\pi}$

$\{ e^{ikx},\ k = 0, \pm 1, \pm 2, ... \}$

If $f \in S_{2\pi}$, $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$, this is fourier series

For f to be real-valued: $a_{-k} = a_k^*$

$a_k e^{ikx} + a_{-k} e^{-ikx} = a_k e^{ikx} + (a_k e^{ikx})^* = 2\,\mathrm{Re}[a_k e^{ikx}]$

- this is fourier series. iit is a lot easier to compt with complex numbers.

$f(x) = a_0 + \left[\sum_{k=1}^{\infty} a_k e^{ikx} + cc. \right]$, a_k are usually complex; cc=complex conj

- we can think of a_k as component
- inf *initebasis*?

$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx} = 0$ if all $a_k = 0$

`Theorem:` $f(x) \equiv 0$ iff $a_{k=0}$ for all k.

Dot Product (Inner product)	$< f, g > \equiv \int_0^{2\pi} f(x) g^*(x) dx$

$< f, f > = \int_0^{2\pi} |f(x)|^2 dx > 0$ unless $f(x) \equiv 0$

Def: f & g are orthogonal if $< f, g > = 0$

- Stef chose orthogonal basis before.

Functions e^{ikx} are orthogonal

Proof: $< e^{ikx}, e^{ilx} >$

$= \int_0^{2\pi} e^{ikx} (e^{ilx})^* dx$ helps to work with exponentials since we can combine them

$= \int_0^{2\pi} e^{ikx} e^{-ilx} dx = \int_0^{2\pi} e^{i(k-l)x} dx$

Suppose $k \neq l$

$\int_0^{2\pi} e^{i(k-l)x} dx = \frac{1}{i(k-l)} e^{i(k-l)x} \Big|_0^{2\pi}$

$= \frac{1}{i(k-l)} [e^{i(k-l)2\pi} - 1]$, goes to zero b/c it is periodic, =1

$= 0$

$k = l$ case

$< e^{ikx}, e^{ilx} > = \int_0^{2\pi} 1 dx = 2\pi$

- *wehave* an orthog basis,
- the norm squared = 2

Theorem:

$f(x)=\sum_{k=-\infty}^{\infty} a_k e^{ikx}, \quad a_k =$ Fourier components

$<f, e^{ilx}>$

$=\sum_{k=-\infty}^{\infty} a_k < e^{ikx}, e^{ilx} >$

$=a_l 2\pi$

$a_l = \frac{1}{2\pi} <f, e^{ilx}>$

$= \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ilx} dx, \ \ l = 0, \pm 1, \pm 2,$

• above = 0 when

• we are projecting the function onto one of th basiss vector,

• it picks out that 1 F coeef.

•*wehae* 2 reps of the funct: 1) 2) coef

$$\left.\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} a_k e^{ikx} \\ a_k &= \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ikx} dx \end{aligned}\right\}$$

• there is assymetry, inf series on top, integral below

• this is classic Fourier series

•this has taken us from physical space to wavenumber space.

• This is a form of a FT

•$f(x) - - - - - \boxed{f} - - - - - - > \{a_k\}$

phys wavenumber space

$< - - - - - \boxed{f^{-1}}$

• spectral refers to the above coeef

• if we know the coef, we have to go in the other direction, that woul be inverse transform

------

$f(x) \equiv 0$ iff

$a_k = 0$ for all k

$\Rightarrow < f, a_k > = 0$ for all k

•$Genmethod$: \

• tells us we can porb not get exact soln., have to do truncation & $\therefore$

Spectral Method AKA projection method(here applied to lin advection eqn)

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$$

$u(x+2\pi,t)=u(x,t)$ all x,t

$u(x,0)=f(x)$

• *wewantittobe* identically zero; to do that , this statement would hvae to be:

• Equivalent statement:

$$<\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x},e^{ilx}>=0 \quad \text{all } l \,\&\, t$$

• ifthatiszero,thishastobe zero

• if we want function to be zero, than it will be if we can show that alll the inner prod are zero

$u(x,t)=\sum_{k=-\infty}^{\infty} a_k(t)e^{ikx}$ here F coef are functions of time.

$$\frac{\partial u}{\partial x}=\sum_{k=-\infty}^{\infty} ika_k(t)e^{ikx}$$

• Beauty of it: when we diff (can do it term by term), all we do is bring down a factor of ik

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x},$$

$$=\sum_{k=-\infty}^{\infty}\left[\frac{da_k}{dt}+ikca_k(t)\right]e^{ikx}$$

$<\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x},e^{ikx}> \;=2\pi\left[\frac{da_k}{dt}+ilca_l\right]=0$ for all ==l;

• this is the key. proj on basis gives ODE

when we have Fseries, and we project it on lth,

then all we get is the lth ______

• these are supposed to be zero

$$\boxed{\frac{da_l}{dt}+ilca_l=0,\;\; l=0,\pm1,\pm2,...}$$

•this is inf set, but they are decoupled; we can solve each ne separately.

• we only hvae to solve for neg numbers.

• we have an infinite set of ODEs

$->a_l(t)=a_l(0)e^{-ilct}$

$$=>u(x,t)=\sum_{l=-\infty}^{\infty} a_l(0)e^{il(x-ct)}=f(x-ct)$$

•*General* : {fill in from Emerson}

•have to truncate

•