Met 240, Assignment 10
John Noble

1) *Consider the "compressible Boussinesq" equations* [Durran, Eqs. (7.54) – (7.56)]
Complete the following derivation […] of a Helmholtz equation for P^{n+1}

~~~~~~

Rewrite equations (1) – (4) in finite difference form

$$\frac{b^{n+1}-b^{n-1}}{2\Delta t}+N^2w^n=0 \tag{1}$$

$$\frac{u^{n+1}-u^{n-1}}{2\Delta t}+u^n\frac{\partial u^n}{\partial x}+\frac{1}{2}\left(\frac{\partial P^{n+1}}{\partial x}+\frac{\partial P^{n-1}}{\partial x}\right)=0 \tag{2}$$

$$\frac{w^{n+1}-w^{n-1}}{2\Delta t}+u^n\frac{\partial w^n}{\partial x}+\frac{1}{2}\left(\frac{\partial P^{n+1}}{\partial z}+\frac{\partial P^{n-1}}{\partial z}\right)=b \tag{3}$$

$$\frac{P^{n+1}-P^{n-1}}{2\Delta t}+u^n\frac{\partial P^n}{\partial x}+c_s^2\left[\left(\frac{1}{2}\left(\frac{\partial u^{n+1}}{\partial x}+\frac{\partial u^{n-1}}{\partial x}\right)\right)+\left(\frac{1}{2}\left(\frac{\partial w^{n+1}}{\partial z}+\frac{\partial w^{n-1}}{\partial z}\right)\right)\right]=0 \tag{4}$$

Rearrange equation (1) with unknown terms ($n+1$) on LHS

$$b^{n+1}=b^{n-1}-2\Delta t\left(N^2w^n\right) \tag{5}$$

Rearrange equation (2) with unknown terms ($n+1$) on LHS

$$u^{n+1}=u^{n-1}+2\Delta t\left[-\frac{1}{2}\left(\frac{\partial P^{n+1}}{\partial x}+\frac{\partial P^{n-1}}{\partial x}\right)-u^n\frac{\partial u^n}{\partial x}\right]$$

$$u^{n+1}+\Delta t\frac{\partial P^{n+1}}{\partial x}=u^{n-1}-\Delta t\frac{\partial P^{n-1}}{\partial x}-2\Delta t u^n\frac{\partial u^n}{\partial x}$$

$$u^{n+1}+\Delta t\frac{\partial P^{n+1}}{\partial x}=F,\quad \text{where } F=u^{n-1}-\Delta t\frac{\partial P^{n-1}}{\partial x}-2\Delta t u^n\frac{\partial u^n}{\partial x} \tag{6}$$

Rearrange equation (3) with unknown terms ($n+1$) on LHS
~~~~~~

$$w^{n+1} = w^{n-1} + 2\Delta t\left[b - \frac{1}{2}\left(\frac{\partial P^{n+1}}{\partial z} + \frac{\partial P^{n-1}}{\partial z}\right) - u^n \frac{\partial w^n}{\partial x}\right]$$

$$w^{n+1} = w^{n-1} + \left[2\Delta tb - 2\Delta t\,\frac{1}{2}\left(\frac{\partial P^{n+1}}{\partial z} + \frac{\partial P^{n-1}}{\partial z}\right) - 2\Delta tu^n \frac{\partial w^n}{\partial x}\right]$$

$$w^{n+1} = w^{n-1} + 2\Delta tb - \Delta t\left(\frac{\partial P^{n+1}}{\partial z} + \frac{\partial P^{n-1}}{\partial z}\right) - 2\Delta tu^n \frac{\partial w^n}{\partial x}$$

$$w^{n+1} + \Delta t\,\frac{\partial P^{n+1}}{\partial z} = w^{n-1} + 2\Delta tb - \Delta t\,\frac{\partial P^{n-1}}{\partial z} - 2\Delta tu^n \frac{\partial w^n}{\partial x}$$

$$w^{n+1} + \Delta t\,\frac{\partial P^{n+1}}{\partial z} = G,\ \text{ where }\ w^{n-1} + 2\Delta tb - \Delta t\,\frac{\partial P^{n-1}}{\partial z} - 2\Delta tu^n \frac{\partial w^n}{\partial x} \tag{7}$$

Rearrange equation (4) with unknown terms ($n + 1$) on LHS

$$P^{n+1} = P^{n-1} + 2\Delta t\left[-c_s^2\left[\left(\frac{1}{2}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^{n-1}}{\partial x}\right)\right) + \left(\frac{1}{2}\left(\frac{\partial w^{n+1}}{\partial z} + \frac{\partial w^{n-1}}{\partial z}\right)\right)\right] - u^n \frac{\partial P^n}{\partial x}\right]$$

$$P^{n+1} = P^{n-1} - 2\Delta tc_s^2\left[\left(\frac{1}{2}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^{n-1}}{\partial x}\right)\right) + \left(\frac{1}{2}\left(\frac{\partial w^{n+1}}{\partial z} + \frac{\partial w^{n-1}}{\partial z}\right)\right)\right] - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} = P^{n-1} - 2\Delta tc_s^2\left[\left(\frac{1}{2}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^{n-1}}{\partial x}\right)\right) + \left(\frac{1}{2}\left(\frac{\partial w^{n+1}}{\partial z} + \frac{\partial w^{n-1}}{\partial z}\right)\right)\right] - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} = P^{n-1} - 2\Delta tc_s^2\left(\frac{1}{2}\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^{n-1}}{\partial x}\right)\right) - 2\Delta tc_s^2\left(\frac{1}{2}\left(\frac{\partial w^{n+1}}{\partial z} + \frac{\partial w^{n-1}}{\partial z}\right)\right) - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} = P^{n-1} - \Delta tc_s^2\left(\frac{\partial u^{n+1}}{\partial x} + \frac{\partial u^{n-1}}{\partial x}\right) - \Delta tc_s^2\left(\frac{\partial w^{n+1}}{\partial z} + \frac{\partial w^{n-1}}{\partial z}\right) - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} = P^{n-1} - \Delta tc_s^2\,\frac{\partial u^{n+1}}{\partial x} - \Delta tc_s^2\,\frac{\partial u^{n-1}}{\partial x} - \Delta tc_s^2\,\frac{\partial w^{n+1}}{\partial z} - \Delta tc_s^2\,\frac{\partial w^{n-1}}{\partial z} - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} + \Delta tc_s^2\,\frac{\partial u^{n+1}}{\partial x} + \Delta tc_s^2\,\frac{\partial w^{n+1}}{\partial z} = P^{n-1} - \Delta tc_s^2\,\frac{\partial u^{n-1}}{\partial x} - \Delta tc_s^2\,\frac{\partial w^{n-1}}{\partial z} - 2\Delta tu^n \frac{\partial P^n}{\partial x}$$

$$P^{n+1} + \Delta tc_s^2\,\frac{\partial u^{n+1}}{\partial x} + \Delta tc_s^2\,\frac{\partial w^{n+1}}{\partial z} = H,\quad \text{where } H = P^{n-1} - \Delta tc_s^2\,\frac{\partial u^{n-1}}{\partial x} - \Delta tc_s^2\,\frac{\partial w^{n-1}}{\partial z} - 2\Delta tu^n \frac{\partial P^n}{\partial x} \tag{8}$$

Rewrite (8)

$$P^{n+1} + \Delta tc_s^2 \nabla u^{n+1} = H \tag{9}$$

Take $\frac{\partial}{\partial x}(6)$

$$u^{n+1} + \Delta t\frac{\partial P^{n+1}}{\partial x} = F, \quad \text{where } F = u^{n-1} - \Delta t\frac{\partial P^{n-1}}{\partial x} - 2\Delta t u^n\frac{\partial u^n}{\partial x}$$

$$\frac{\partial}{\partial x}\left[u^{n+1} + \Delta t\frac{\partial P^{n+1}}{\partial x}\right] = \frac{\partial}{\partial x}\left[u^{n-1} - \Delta t\frac{\partial P^{n-1}}{\partial x} - 2\Delta t u^n\frac{\partial u^n}{\partial x}\right]$$

$$\frac{\partial u^{n+1}}{\partial x} + \Delta t\frac{\partial^2 P^{n+1}}{\partial x^2} = \frac{\partial u^{n-1}}{\partial x} - \Delta t\frac{\partial^2 P^{n-1}}{\partial x^2} - 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x} + u^n\frac{\partial^2 u^n}{\partial x^2}\right]$$

$$\frac{\partial u^{n+1}}{\partial x} = \frac{\partial u^{n-1}}{\partial x} - \Delta t\frac{\partial^2 P^{n-1}}{\partial x^2} - 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x} + u^n\frac{\partial^2 u^n}{\partial x^2}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial x^2} \tag{10}$$

Take $\frac{\partial}{\partial z}(7)$

$$\frac{\partial}{\partial z}\left[w^{n+1} + \Delta t\frac{\partial P^{n+1}}{\partial z}\right] = \frac{\partial}{\partial z}\left[w^{n-1} + 2\Delta t b - \Delta t\frac{\partial P^{n-1}}{\partial z} - 2\Delta t u^n\frac{\partial w^n}{\partial x}\right]$$

$$\frac{\partial}{\partial z}w^{n+1} + \Delta t\frac{\partial^2 P^{n+1}}{\partial z^2} = \frac{\partial}{\partial z}w^{n-1} + 2\Delta t\frac{\partial}{\partial z}b - \Delta t\frac{\partial^2 P^{n-1}}{\partial z^2} - 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x} + u^n\frac{\partial w^n}{\partial z\partial x}\right]$$

$$\frac{\partial}{\partial z}w^{n+1} = \frac{\partial}{\partial z}w^{n-1} + 2\Delta t\frac{\partial}{\partial z}b - \Delta t\frac{\partial^2 P^{n-1}}{\partial z^2} - 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x} + u^n\frac{\partial w^n}{\partial z\partial x}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial z^2} \tag{11}$$

Substitute (10) and (11) into (8)

$$\begin{aligned} P^{n+1} &+ \Delta t c_s^2\left[\frac{\partial u^{n-1}}{\partial x} - \Delta t\frac{\partial^2 P^{n-1}}{\partial x^2} - 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x} + u^n\frac{\partial^2 u^n}{\partial x^2}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial x^2}\right] + \\ &\Delta t c_s^2\left[\frac{\partial}{\partial z}w^{n-1} + 2\Delta t\frac{\partial}{\partial z}b - \Delta t\frac{\partial^2 P^{n-1}}{\partial z^2} - 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x} + u^n\frac{\partial w^n}{\partial z\partial x}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial z^2}\right] = H \end{aligned} \tag{12}$$

Rewrite (12)

$$\begin{aligned} P^{n+1} &+ \Delta t c_s^2\left[\frac{\partial u^{n-1}}{\partial x} - \Delta t\frac{\partial^2 P^{n-1}}{\partial x^2} - 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x} + u^n\frac{\partial^2 u^n}{\partial x^2}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial x^2}\right] + \\ &\Delta t c_s^2\left[\frac{\partial}{\partial z}w^{n-1} + 2\Delta t\frac{\partial}{\partial z}b - \Delta t\frac{\partial^2 P^{n-1}}{\partial z^2} - 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x} + u^n\frac{\partial w^n}{\partial z\partial x}\right] - \Delta t\frac{\partial^2 P^{n+1}}{\partial z^2}\right] = H \end{aligned}$$

$$P^{n+1}+\Delta t c_s^2\frac{\partial u^{n-1}}{\partial x}-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial x^2}-\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x}+u^n\frac{\partial^2 u^n}{\partial x^2}\right]-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n+1}}{\partial x^2}+$$
$$\Delta t c_s^2\frac{\partial}{\partial z}w^{n-1}+\Delta t c_s^2 2\Delta t\frac{\partial}{\partial z}b-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial z^2}-\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x}+u^n\frac{\partial w^n}{\partial z\partial x}\right]-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n+1}}{\partial z^2}=H$$

Rewrite in Helmholtz form

$$P^{n+1}+\Delta t c_s^2\frac{\partial u^{n-1}}{\partial x}-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial x^2}-\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x}+u^n\frac{\partial^2 u^n}{\partial x^2}\right]-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n+1}}{\partial x^2}+$$
$$\Delta t c_s^2\frac{\partial}{\partial z}w^{n-1}+\Delta t c_s^2 2\Delta t\frac{\partial}{\partial z}b-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial z^2}-\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x}+u^n\frac{\partial w^n}{\partial z\partial x}\right]-\Delta t c_s^2\Delta t\frac{\partial^2 P^{n+1}}{\partial z^2}=H$$

$\Rightarrow$

$$-\Delta t c_s^2\Delta t\left(\frac{\partial^2 P^{n+1}}{\partial x^2}+\frac{\partial^2 P^{n+1}}{\partial z^2}\right)+P^{n+1}=H-\Delta t c_s^2\frac{\partial u^{n-1}}{\partial x}+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial x^2}+\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x}+u^n\frac{\partial^2 u^n}{\partial x^2}\right]-$$
$$\Delta t c_s^2\frac{\partial}{\partial z}w^{n-1}-\Delta t c_s^2 2\Delta t\frac{\partial}{\partial z}b+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial z^2}+\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x}+u^n\frac{\partial w^n}{\partial z\partial x}\right]$$

$$\Rightarrow\nabla^2 P^{n+1}-\frac{1}{\Delta t c_s^2\Delta t}P^{n+1}=-\frac{1}{\Delta t c_s^2\Delta t}\left[\begin{array}{l}H-\Delta t c_s^2\frac{\partial u^{n-1}}{\partial x}+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial x^2}+\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x}+u^n\frac{\partial^2 u^n}{\partial x^2}\right]-\\ \Delta t c_s^2\frac{\partial}{\partial z}w^{n-1}-\Delta t c_s^2 2\Delta t\frac{\partial}{\partial z}b+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial z^2}+\\ \Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x}+u^n\frac{\partial w^n}{\partial z\partial x}\right]\end{array}\right]$$

Simplify

$$\nabla^2 P^{n+1}+\lambda P^{n+1}=\lambda\left[\begin{array}{l}H-\Delta t c_s^2\frac{\partial u^{n-1}}{\partial x}+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial x^2}+\Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial x}\frac{\partial u^n}{\partial x}+u^n\frac{\partial^2 u^n}{\partial x^2}\right]-\\ \Delta t c_s^2\frac{\partial}{\partial z}w^{n-1}-\Delta t c_s^2 2\Delta t\frac{\partial}{\partial z}b+\Delta t c_s^2\Delta t\frac{\partial^2 P^{n-1}}{\partial z^2}+\\ \Delta t c_s^2 2\Delta t\left[\frac{\partial u^n}{\partial z}\frac{\partial w^n}{\partial x}+u^n\frac{\partial w^n}{\partial z\partial x}\right]\end{array}\right]$$

where $\lambda=-\frac{1}{c_s^2\Delta t^2}$