Gravity : $\vec{F} = \frac{Gm_e m_s}{r^2}$; **Centrifugal :** $\vec{F_c} = \frac{m_s v^2}{r}$; Geopotential: $d\Phi\left[\text{J/kg or m}^2/\text{s}^2\right] \equiv gdz = -\alpha dp$

Momentum, Linear : $\vec{\boldsymbol{p}} = m\vec{\boldsymbol{V}}$; **Angular :** $\vec{\boldsymbol{L}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{p}}$ $\quad L = mVR$

Coriolis parmater : $f_c \equiv 2\Omega \sin\phi \left[\text{s}^{-1}\right]$; $-f_c v \cong -\frac{1}{\rho}\frac{\partial p}{\partial x}$; $f_c u \cong -\frac{1}{\rho}\frac{\partial p}{\partial y}$

Hydrostatic balance : $\frac{\partial p}{\partial z} = -\rho g$; $\quad \frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$; **PGF :** $-\frac{1}{\rho}\frac{\partial p}{\partial x}\left[\text{m s}^{-2}\right]$

$$z_1 = \frac{R\overline{T}}{g}\ln\left(\frac{p_s}{p_1}\right)$$

Geostrophic Wind : $\vec{\boldsymbol{V}}_g \equiv \hat{\boldsymbol{k}} \times \frac{1}{\rho f_c}\vec{\nabla} p$

Rossby Number : $R_0 \equiv \left(U^2/L\right)/\left(f_c U\right) \Rightarrow U/\left(f_c L\right)$

IGL : $pV = mRT \Rightarrow p = \rho RT$

Continuity Equation : $\frac{1}{\rho}\frac{d\rho}{dt} = -\vec{\nabla} \bullet \vec{V}$; $\quad \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \varpi}{\partial p} = 0$

Divergence : $\vec{V} = ax\hat{i} + by\hat{j}, (a > 0, b > 0)$, $\vec{\nabla} \bullet \vec{V} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = a + b > 0$

Thermodynamic : $de = dq + dw$; $dq = du - dw$; $dq = c_v dT + pd\alpha$

$$c_p \frac{dT}{dt} - \alpha\varpi = \frac{dq}{dt}$$

$$\frac{d\vec{\boldsymbol{V}}_a}{dt} = -\frac{1}{\rho}\nabla p + \vec{\boldsymbol{g}}_a + \vec{\boldsymbol{F}}$$

$$\frac{d\vec{\boldsymbol{V}}_r}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{V}}_r + \vec{\boldsymbol{g}}_a + \Omega^2\vec{\boldsymbol{R}} + \vec{\boldsymbol{F}}; \quad \vec{\boldsymbol{g}} = \vec{\boldsymbol{g}}_a + \Omega^2\vec{\boldsymbol{R}}$$

$$\frac{d\vec{\boldsymbol{V}}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{V}} + \vec{\boldsymbol{g}} + \vec{\boldsymbol{F}}$$

$\frac{Du}{Dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$ OR approximation: $\frac{Du}{Dt} - = fv - \frac{1}{\rho}\frac{\partial p}{\partial x}$

$\frac{Dv}{Dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$ OR approximation: $\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}$

$$\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz} \Rightarrow \frac{\partial p}{\partial z} = -\rho g$$

$$p = \rho RT$$

$$\frac{d\rho}{dt} = -\rho\vec{\nabla} \bullet \vec{V}$$

$$\frac{dq}{dt} = c_p\frac{dT}{dt} - \alpha\frac{dp}{dt}$$

Total Derivative : $\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \vec{V} \bullet \vec{\nabla} T \Rightarrow \frac{Dt}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$

where $u = \frac{dx}{dt}$; $v = \frac{dy}{dt}$; $w = \frac{dz}{dt}$

Gradients : $\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$; $\quad \vec{\nabla}(\text{constant}) = 0$

$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial\phi}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial\phi}{\partial z}\hat{\boldsymbol{k}} \Rightarrow \vec{\nabla}(\text{scalar}) = \text{vector}$

$\vec{\nabla} \bullet \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \Rightarrow \vec{\nabla} \bullet (\text{vector}) = \text{scalar}$

$\vec{\nabla} \bullet \vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial^2 z}$

Curl : $\vec{\nabla} \times \vec{V}$: $\vec{\nabla} \times \vec{V} = 0 \Rightarrow \vec{V}$ is irrotational, no vorticity;

$\vec{\nabla} \times \vec{V} > 0 \Rightarrow$ cyclonic (NH CCW); $< 0 \Rightarrow$ Anticyc (CW)

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\boldsymbol{i}} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\boldsymbol{k}}$$

Vorticity : $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \hat{\boldsymbol{k}} \bullet \left(\vec{\nabla} \times \vec{V}\right)$

Divergence : $\vec{\nabla} \bullet \vec{V} = 0 \Rightarrow \vec{V}$ is non divergent;

$+ \Rightarrow$ divergence; $- \Rightarrow$ convergence

$$\vec{\nabla} \bullet \vec{V} = \left\langle \frac{\partial}{\partial x}\hat{\boldsymbol{i}}, \frac{\partial}{\partial y}\hat{\boldsymbol{j}}, \frac{\partial}{\partial z}\hat{\boldsymbol{k}} \right\rangle \bullet \left\langle u\hat{\boldsymbol{i}}, v\hat{\boldsymbol{j}}, w\hat{\boldsymbol{k}} \right\rangle = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}; \quad \vec{\nabla}_H \bullet \vec{V}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Dimensions and units

Mass	M	Kg
Length	L	m
Time	T=L/U	s
Temp.	K	K
Velocity	L/T = U	m/s
Accel	L/T^2	m/s^2
Force	ML/T^2	$Kg\,m/s^2$ = N (Newton)
Pressure		$Kg \cdot m^{-1}s^{-2} = N \cdot m^{-2} = Pa$

Temperature : $(9/5 \times °C) + 32 = °F$; $(°F - 32) \times 5/9 = °C$; K=°C +273.15

Area : $1\ \text{cm}^2 = 10^{-4}\ \text{m}^2 \Leftrightarrow 1\ \text{m}^2 = 10^4\ \text{cm}^2$

Volume : $1\ \text{liter} = 10^3\text{cm}^3 = 10^{-3}\text{m}^3$; $\quad 1\ \text{m}^3 = 10^6\text{cm}^3 \Leftrightarrow 1\text{cm}^3 = 10^{-6}\ \text{m}^3$

Pressure : 1atm = 1013.25mb = 1013.25hPa = 101.325kPa = 1.01325×10^5Pa

1 hPa = 100 Pa

Density : $1\ \text{gm cm}^{-3} = 1000\ \text{kg m}^{-3}$ $\quad \rho_0 = 1.225\ \text{kg} \cdot \text{m}^{-3}$

$c_p = 1004\ \text{J kg}^{-1}\ \text{K}^{-1} = 1.00464\ \text{J gm}^{-1}\ \text{K}^{-1}$ (const for IG)

$c_v = 717\ \text{J kg}^{-1}\ \text{K}^{-1} = 0.7176\ \text{J gm}^{-1}\ \text{K}^{-1}$, $c_v = \left(\frac{\partial u}{\partial T}\right)_\alpha$ (for any substance)

$a = 6.37 \times 10^6\,\text{m}$

$R_d = 287.053\ \text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$

Properties of vectors

$\vec{a} \bullet \vec{b} = 0 \Rightarrow \vec{a}\ \&\ \vec{b}$ are orthogonal

$\vec{a} \bullet \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta = ab\cos\theta \quad (0 \le \theta \le \pi)$

The length/magnitude of the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$\vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c}$

$\vec{a} + \vec{0} = \vec{a}$

$\vec{a} + \left(-\vec{a}\right) = \vec{0}$

$c\left(\vec{a} + \vec{b}\right) = c\vec{a} + c\vec{b}$

$(c + d)\vec{a} = c\vec{a} + d\vec{a}$

$(cd)\vec{a} = c\left(d\vec{a}\right)$

$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \bullet \vec{c}\right)\vec{b} - \left(\vec{a} \bullet \vec{b}\right)\vec{c}$

$\left|\vec{a} \times \vec{b}\right| = \|\vec{a}\|\|\vec{b}\|\sin\theta$

$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$

Vector multiplication with Scalar

$c\vec{a} = c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$

♦ 2 vectors are $\parallel$ if & only if they are scalar multiples of eachother $\left(-2\vec{a} = \vec{b}\right)$

Unit Vectors : $\vec{u} = \frac{1}{\|\vec{a}\|}\vec{a} = \frac{\vec{a}}{\|\vec{a}\|}$

$\vec{i} = \langle 1,0,0 \rangle \quad \vec{j} = \langle 0,1,0 \rangle \quad \vec{k} = \langle 0,0,1 \rangle$

Ex: $\langle 1,-2,6 \rangle = \vec{i} - 2\vec{j} + 6\vec{k}$

Dot Product (Scalar Product, Inner Product)

Definition: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$

then the dot product of $\vec{a}$ & $\vec{b}$ is the scalar:

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Properies of the Dot Product

$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$\left(c\vec{a}\right) \cdot \vec{b} = c\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \cdot \left(c\vec{b}\right)$

$\vec{0} \cdot \vec{a} = 0$

Dot Product Theorem 1 & Corollary

If θ is the angle between the vectors $\vec{a}$ & $\vec{b}$, then:

$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$

Orthagonality 2 vectors $\vec{a}$ & $\vec{b}$ are orthagonal if & only if: $\vec{a} \cdot \vec{b} = 0$

Cross Product (Vector Product)

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Orthagonality Vector $\vec{a} \times \vec{b}$ is orthag to both $\vec{a}\ \&\ \vec{b}$

Cross Product Theorem & Corollary

If θ is the angle between the $\vec{a}$ & $\vec{b}$,

(so $0 \le \theta \le \pi$), then: $\left|\vec{a} \times \vec{b}\right| = \|\vec{a}\|\|\vec{b}\|\sin\theta$

Two nonzero vectors $\vec{a}$ & $\vec{b}$ are parallel if & only if: $\vec{a} \times \vec{b} = \vec{0}$

Theorem : If $\vec{a}, \vec{b}, \&\ \vec{c}$ are vectors and d is a scalar, then: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\left(d\vec{a}\right) \times \vec{b} = d\left(\vec{a} \times \vec{b}\right) = \vec{a} \times \left(d\vec{b}\right)$

$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\left(\vec{a} + \vec{b}\right) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$

$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c}$

BASIC EQUATIONS : Rect height coord (RHC), Isobaric (IC), Natural (NC)

Governing equations in vector form, rectangular & pressure coordinates

RHC	IC	
$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} + \vec{g} + \vec{F}$	$\frac{D\vec{V}}{Dt} = -\vec{\nabla}_p\Phi - 2\Omega\times\vec{V}$	**Horizontal Momentum Eqn.**
$\frac{\partial p}{\partial z} = -\rho g$	$\frac{\partial\Phi}{\partial p} = -\frac{1}{\rho}$	**Hydrostatic balance**
$p = \rho RT$	$p = \rho RT$	**Ideal Gas Law**
$\frac{1}{\rho}\frac{d\rho}{dt} = -\vec{\nabla}\bullet\vec{V}$	$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial\varpi}{\partial p} = 0$	**Continuity Equation**
$c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \frac{dq}{dt}$	$c_p\frac{dT}{dt} - \alpha\varpi = \frac{dq}{dt}$	**Thermodynamic Energy Eqn.**

where: $\left(\frac{d}{dt}\right)_p = \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \varpi\left(\frac{\partial}{\partial p}\right)_p$

where $\varpi = \frac{dp}{dt}$ | Omega vertical motion. p change following the motion. Same as $w = dz/dt$ in RC. ; $f_c = 2\Omega\sin\phi$

$\vec{V}_g \equiv \hat{k}\times\frac{1}{\rho f_c}\vec{\nabla}p$ $\qquad f_c\vec{V}_g = \hat{k}\times\vec{\nabla}_p\Phi$ | **Geostrophic relationship**

PROPERTIES OF THE VELOCITY FIELD (VF)

The kinematic properties of the VF are determined by its divergence & curl, i.e. by differential operators. $\nabla\bullet\vec{V}$ & $\nabla\times\vec{V}$

NATURAL (INTRINSIC) COORDINATE SYSTEM (NC) : Natural coordinates are flow following coordinates used to better understand how vorticity & divergence arise in flows.

Trajectory (Path) : The locus of successive positions of a moving fluid parcel. @ any given instant the velocity vector of the parcel is tangent to the trajectory. Trajectories are lines connecting the positions of a fluid parcel at successive instants in time, ie. the actual path followed by the parcel.

Streamline : A line whose tangent at any point in a fluid is parallel to the instantaneous velocity vector of the fluid at that point (at an instant in time). Points on the streamline are at the same time. NCS is an orthogonal right-handed system.

The wind vector $\vec{V}_H$ defines the unit tangent vector $\hat{t}$ at P. The normal coordinate, n, increases to the left of the wind direction, and with $\hat{n}$. $\hat{n}$ is normal to $\vec{V}_H$ and is positive to the left of flow.

The unit vectors obey the relations: $\hat{t}\times\hat{n} = \hat{k}$, $\hat{n}\times\hat{k} = \hat{t}$, $\hat{k}\times\hat{t} = \hat{n}$ & $\hat{i}\hat{j}\hat{k}$ is $\hat{t}\hat{n}\hat{k}$ in NC.

Sign convention: (applies to both Northern & Southern Hemispheres (NH & SH)

CCW: $d\theta > 0 \Rightarrow K, K_s, R, R_s > 0$ $\qquad$ CW: $d\theta < 0 \Rightarrow K, K_s, R, R_s < 0$

R is positive when when center of curvature is in the positive $\hat{n}$ direction.

Trajectory	Streamline	Remarks
$\frac{d\theta}{ds} = K = \frac{1}{R}$	$\frac{\partial\theta}{\partial s} = K_s = \frac{1}{R_s}$	Curvature of path/trajectory; i.e. change in wind direction downstream along trajectory
$\frac{d\hat{t}}{d\theta} = \hat{n}$	$\frac{\partial\hat{t}}{\partial\theta} = \hat{n}$	
$\frac{d\hat{t}}{ds} = K\hat{n}$	$\frac{\partial\hat{t}}{\partial s} = K_s\hat{n}$	
$\frac{d\hat{t}}{dn} = \frac{d\theta}{dn}\hat{n}$	$\frac{\partial\hat{t}}{\partial n} = \frac{\partial\theta}{\partial n}\hat{n}$	Change in wind direction normal to flow
$\frac{d\hat{t}}{dt} = KV\hat{n}$	$\frac{\partial\hat{t}}{\partial t} = \frac{\partial\theta}{\partial t}\hat{n}$	

Geostrophic flow $\boxed{f_c V_g = -\frac{\partial\Phi}{\partial n}}$ | Straight line flow $\Rightarrow R \to \pm\infty$; $f_c V$ = Coriolis force; $\frac{\partial\Phi}{\partial n} = \frac{1}{\rho}\frac{\partial p}{\partial n}$ = PGF

Inertial flow $\boxed{\frac{V^2}{R} + f_c V = 0}$ $R = -\frac{V}{f}$ | Circular flow paths in the anticyclonic sense: $V > 0, f > 0, R < 0$

Cyclostorphic flow

$\boxed{\frac{V^2}{R} = -\frac{\partial\Phi}{\partial n}}$ | LHS = Centrifugal force; RHS = PGF; Force balance normal to flow direction $\boxed{V = \left(-R\frac{\partial\Phi}{\partial n}\right)^{1/2}}$ | Cyclostrophic wind speed.

Gradient wind Balance $\boxed{\frac{V^2}{R} + f_c V = -\frac{\partial\Phi}{\partial n}}$ $\boxed{V = -\frac{fR}{2} \pm \sqrt{\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}}}$

Around Low p : gradient wind is weaker than geostrophic wind, ie. geostrophic wind is an overestimation. Around High: gradient wind is stronger the geostophic.

DIVERGENCE : $\boxed{\vec{\nabla}_H\bullet\vec{V}_H = \frac{\partial V}{\partial s} + V\frac{\partial\theta}{\partial n}}$ [NC] | This is velocity divergence, not mass divergence: $\nabla\bullet\rho\vec{V}$

1st term = longitudinal divergence, and it is >0 if the wind speed ↑in the downstream direction along the streamlines. 2nd term = transversal divergence, and it is >0 if the streamlines "diverge" in the direction normal to the flow.

Non - divergent flow : 1) it is possible for there to be non-div. flow even if the streamlines seem to indicate divergence or convergence, ie. when the 2 terms above are balanced.

2) If an area does not change it numerical value, ie. A_1=A_2 (although may change shape), then the flow is non-divergent. ∴ *difluence*, spreading out of streamlines, does not by itself imply divergence. Difluence is measured by $V(\partial\theta/\partial n)$ only.

Convergence : $\vec{\nabla}_H\bullet\vec{V}_H < 0 \Rightarrow \partial\theta/\partial n < 0,\ \partial V/\partial s < 0$

Divergence : $\vec{\nabla}_H\bullet\vec{V}_H > 0 \Rightarrow \partial\theta/\partial n > 0,\ \partial V/\partial s > 0$

STREAMFUNCTION : When horizontal flow is such that $\vec{\nabla}_H\bullet\vec{V}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ everywhere, flow is non-divergent. $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ is the requirement for exactness of the differential $vdx - udy$. ∴ $d\psi = vdx - udy = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$. ∴ when $\vec{\nabla}_H\bullet\vec{V}_H = 0$ velocity components can be expressed as $\boxed{u = -\frac{\partial\psi}{\partial y}, v = \frac{\partial\psi}{\partial x}}$ | ψ = streamfunction $[L^2/T]$

For nondivergent flow, the velocity field can be represented by SF alone.

$\boxed{\vec{V}_H = u\hat{i} + v\hat{j} = -\frac{\partial\psi}{\partial y}\hat{i} + \frac{\partial\psi}{\partial x}\hat{j}}$ OR $\boxed{\vec{V}_H = \hat{k}\times\nabla_H\psi}$

$\boxed{\vec{\nabla}_H\bullet\vec{V}_H = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial y\partial x} = 0}$ $\boxed{\zeta = \hat{k}\bullet\vec{\nabla}_H\times\vec{V}_H = \nabla_H^2\psi}$

The isopleths of ψ are streamlines, & are always tangent to the instantaneous wind vector $\vec{V}_H$. However the representation of $\vec{V}_H$ by ψ alone is only possible if $\vec{\nabla}_H\bullet\vec{V}_H = 0$. Streamlines show direction of flow and the speed is inversely proportional to the spacing of the streamlines.

VORTICITY : A vector measure of the tendency of a fluid parcel to rotate about an axis through its center. Vorticity is the curl of the velocity field. It is an extension of the concept of the angular velocity of a fluid parcel as it rotates about some axis.

$\vec{q} = \vec{\nabla}\times\vec{V}$; $\vec{q}$ is a 3D vector; $\vec{q}$ defines a vector field.

We are primarily interested in the tendency of fluid parcels to rotate about their local verticals:

$\boxed{\zeta = \hat{k}\bullet\left(\vec{\nabla}\times\vec{V}\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$ | Rectangular coordinates; ζ is relative vort since $\vec{V}$ is the relative wind (to rotating Earth)

$\zeta > 0 \Rightarrow$ CCW rotation (NH: cyclonic, around low p system; SH: anticyclonic)

$\zeta < 0 \Rightarrow$ CW rotation (NH: anticyclonic, around high p system; SH: cyclonic)

Rotation means rotation about an axis through its center-of-mass. There are circular flows for which $\zeta = 0$, and there are straight-line flows for which $\zeta \neq 0$

$\boxed{\vec{V}_H = V\hat{t}}$ $\boxed{\vec{\nabla}_H = \hat{t}\frac{\partial}{\partial s} + \hat{n}\frac{\partial}{\partial n}}$ | $\frac{\partial}{\partial s}$ means diff in the downstream direction, $\frac{\partial}{\partial n} \Rightarrow$ cross-steam

$\boxed{\zeta = VK_s - \frac{\partial V}{\partial n} = \frac{V}{R_s} - \frac{\partial V}{\partial n}}$ | Vertical component of relative Vorticity in natural coordinates; 1st term is curvature term, 2nd is shear term

∴ ζ is due to the superposition of 2 effects: one is the effect of the *streamline curvature*, the other is the effect of the *speed shear* normal to the flow.

∴Straight parallel flow can possess vorticity. The flow has no curvature, but if there is a variation of speed normal to the direction of flow, ζ will not be zero.

∴Curved flow may be irrotational (ζ=0) when the curvature effect is exactly balanced by shear

Low p sys (typical) in NH : $K_s > 0;\ \vec{V}K_s = \frac{\vec{V}}{R_s} > 0;\ \frac{\partial\vec{V}}{\partial n} < 0;\ \Rightarrow \zeta > 0$ | The sign of the curvature dominates, ∴ $\zeta > 0$

High p sys (typical) in NH : $K_s < 0;\ \vec{V}K_s = \frac{\vec{V}}{R_s} < 0;\ \frac{\partial\vec{V}}{\partial n} > 0;\ \Rightarrow \zeta < 0$ | The sign of the curvature dominates, ∴ $\zeta < 0$

ΔTEMPERATURE

$\frac{dT}{dy}$: $\Delta y = R\cdot\sin(\Delta\phi)$; $\frac{dT}{dx}$: $\Delta x = R\cos\phi\sin(\Delta\lambda)$ | ϕ = latitude, λ=longitude

Sector of a circle: $s = r\theta$ (θ in rads)

THERMAL WIND (TW):

$U_g(p_1) - U_g(p_0) = -\frac{R}{f_c}\frac{\partial\bar{T}}{\partial y}\bigg|_p \ln\left(\frac{p_0}{p_1}\right)$ | LHS = U_T; NH: $f_c > 0$; $\partial T/\partial y$ <0 going poleward. ∴ $U_T > 0$

$V_g(p_1) - V_g(p_0) = \frac{R}{f_c}\frac{\partial\bar{T}}{\partial x}\bigg|_p \ln\left(\frac{p_0}{p_1}\right)$

$\Omega = 2\pi/86400\text{sec} = 7.27\times10^{-5}\text{sec}^{-1}$

VELOCITY POTENTIAL (ϕ): A scalar function whose gradient is proportional to $\vec{V}_H$

$\phi = [L^2/T]$ If $\zeta = \hat{k}\bullet\left(\vec{\nabla}_H\times\vec{V}_H\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \Rightarrow$ **Irrotational flow**

u & v are no longer independent & must satisfy: $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$. ∴ the spatial distribution of the wind field must be such that the shear & curvature effects balance exactly.

$\vec{V}_H = \vec{\nabla}_H\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$ so that $u = \frac{\partial\phi}{\partial x}$ & $v = \frac{\partial\phi}{\partial y}$

$\boxed{\zeta = \hat{k}\bullet\left(\vec{\nabla}_H\times\vec{V}_H\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0}$ $\boxed{\vec{\nabla}_H\bullet\vec{V}_H = \nabla_H^2\phi}$

∴ When the flow is irrotational, it can be represented by Velocity Potential (ϕ) alone.

The isopleths of ϕ, equipotential lines, are ⊥ to the flow when $\vec{V}_H$ is given in terms of ϕ alone. Negative V.P. centers ⇒ regions of large-scale divergence. Positive VP ⇒ conv.

Non - Divergent & Irrotational flow : A special class of flow which can be represented either in terms of a SF alone or a VP alone. ψ = const everywhere ⊥ to ϕ = constant

Horizontal Equations of motion in Natural Coordinates

$\frac{\partial V}{\partial t} = -\frac{\partial\Phi}{\partial s}$ | Along the flow. $\qquad \frac{V^2}{R} + f_c V = -\frac{\partial\Phi}{\partial n}$ | Perpendicular to the flow. {Centrifugal + Coriolis = PGF}

WAVES

Properties of Mechanical Waves (From "University Physics")

- Transverse waves: The elements in the medium vibrate perpendicular to the direction that the wave travels.
- Longitudinal waves: The elements in the medium vibrate parallel to the direction that the wave travels.
- A harmonic wave and an impulsive disturbance travel at the same speed through a medium.
- The wave speed is independent of the amplitude of the wave.
- The wave speed is independent of the frequency of the disturbance.
- The speed v, frequency f, and wavelength λ are related by the equation

$\boxed{v\left[\frac{\text{m}}{\text{s}}\right] = \lambda[\text{m}]\cdot f\left[\text{s}^{-1}\right]}$ Wavelength-frequency relation

Amplitude: max. magnitude of displacement from equilibrium

Wavelength: Distance from one crest to the next.

Period: $T\left[\frac{\text{s}}{\text{cyc}}\right]=\frac{1}{f}$; ie. the time for one cycle.

Frequency: $f\left[\frac{\text{cyc}}{\text{s}}=\text{Hz}\right]=\frac{1}{T}$; # of cycles of oscillation that occur/sec

Angular freq: $\omega\left[\frac{\text{rad}}{\text{s}}\right]=2\pi f=\frac{2\pi}{T}$; | # of radians/sec this corresponds to on the reference circle.

We may regard the number 2π as having units of rad/cycle.

Simple harmonic motion (SHM):

$$a_x=\frac{d^2x}{dt^2}=-\omega^2 x=\frac{k}{m}x;\quad \boxed{\omega=\sqrt{\frac{k}{m}}}$$

$k\left[\frac{\text{N}}{\text{m}}\right]$ = force constant (always >0)

$$f=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m}};\qquad T=\frac{1}{f}=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{k}{m}}$$

Periodic waves

$v=\frac{\lambda}{T}\Rightarrow\boxed{v=\lambda f}$ | The wave pattern travels w/ constant speed v & advances a distance of 1 wavelength λ in a time interval of 1 period T

When a sinusoidal wave passses through a medium, every particle in the medium oscillates w / simple harmonic motion w / the same amplitude & frequency. The frequency is a property of the *entire* periodic wave because all points on the string oscillate with the same frequency.

Wave Functions, eg. $y=y(x,t)$

Phase angle ϕ indicates the initial position of the vibrating object - its relative position in the vibrational motion at time zero.

Sinusoidal wave moving in the positive x direction:

$$y(x,t)=A\cos\left[\omega\left(\frac{x}{v}-t\right)\right]\Leftrightarrow y(x,t)=A\cos\left[2\pi f\left(\frac{x}{v}-t\right)\right]\Leftrightarrow$$

$$y(x,t)=A\cos\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right]\Leftrightarrow\boxed{y(x,t)=A\cos(kx-\omega t)}$$

by substituting $f=\frac{\omega}{2\pi}$ into $v=\lambda f\Rightarrow\boxed{\omega=vk}$

Wave number: $k=\frac{2\pi}{\lambda}\left[\text{m}^{-1}\right]$

Sinusoidal wave moving in the negative x direction:

$$y(x,t)=A\cos\left[2\pi f\left(\frac{x}{v}+t\right)\right]\Leftrightarrow y(x,t)=A\cos\left[2\pi\left(\frac{x}{\lambda}+\frac{t}{T}\right)\right]\Leftrightarrow$$

$$\boxed{y(x,t)=A\cos(kx+\omega t)}$$

Sinusoidal wave moving in negative or positive x direction:

$\boxed{y(x,t)=A\cos(kx\pm\omega t)}$ where $(kx\pm\omega t)$ is called the **phase.** It plays the role of an angular quantity [rad], and its value for any values of x and t determines what part of the sinusoidal cycle is occurring at a particular point and time.

Phase speed v**:** For a wave moving in the pos. x direction, $kx-\omega t=$ constant. Taking the derivative w/ respect to t produces:

$$\frac{dx}{dt}=\frac{\omega}{k}=v$$

Standing waves (SW)

The distance between successsive nodes (or successive antinodes) is $\lambda/2$

The distance between a node and the adjacent antinode is $\lambda/4$

N = nodes. The 2 waves have the following charactarisitics at nodes:

- Exactly out of phase $\Rightarrow$ total wave at that instant is zero
- Resultant displacement is always zero, displ. is *equal* & *opposite*
- $\therefore$ they cancel eachother out $\Rightarrow$ *destructive interference*

A = antinodes. The 2 waves have the following charactaristics:

- Displacements are always identical
- Resultant displacement is always twice the ampl. of each indiv.
- $\Rightarrow$ *constructive interference*
- When they are exactly in phase, resultant displacement is @ maximum.

We can derive a function for the standing wave by adding the functions for 2 waves with equal amplitude, period, & wavelength traveling in opposite directions. We noted that the wave reflected from a fixed end is inverted, so we give a neg. sign to one og the waves.

$y_1(x,t)=-A\cos(kx+\omega t)$ {incident wave traveling left}

$y_2(x,t)=A\cos(kx-\omega t)$ {reflected wave traveling right}

We can rewrite each of the cosine terms by using the identities for the cosine of the sum & difference of 2 angles:

$\boxed{\cos(a\pm b)=\cos a\cos b\pm\sin a\sin b}$ Applying these & combining:

$\boxed{y(x,t)=(2A\sin kx)\sin\omega t}$ Standing wave on a string, fixed end @ $x=0$

The positions of nodes for standing waves are the points for which $\sin kx=0$, so the displacement is always zero. This occurs when

$kx=0,\pi,2\pi,3\pi,\ldots$ or using $k=\frac{2\pi}{\lambda}$, $x=0,\frac{\pi}{k},\frac{2\pi}{k},\frac{3\pi}{k},\ldots$

$x=0,\frac{\lambda}{2},\frac{2\lambda}{2},\frac{3\lambda}{2},\ldots$

Standing wave:
- wave shape stays in same position
- oscillating up & down as described by the $\sin\omega t$ factor.
- each point undergoes SHM, but all points between nodes osc. in phase
- doesn't transfer energy from 1 end to the other $\Rightarrow$ avg transfer rate = 0 the 2 waves indiv. carry = amounts of power in opposite directions.

Traveling wave:
- phase differences between oscillations of adjacent points
- does transfer energy

WAVES - Met 121B

$\boxed{k=\frac{2\pi a\cos\varphi}{\lambda}}$ | Wavenumber: length of the spatial domain divided by the wavelength

$v\left[\frac{\text{m}}{\text{s}}\right]=\lambda[\text{m}]\times f\left[\text{s}^{-1}\right]$ Wave speed = wavelength $\times$ frequency

$\boxed{\Phi(x,y,t)=\Phi_0+\Phi'\sin[k(x-ct)]\cos(ly)}$ | Φ' = amplitude, c = phase speed; k = wave number in x direction; l = wave number in y direction

WAVES

Wavenumber: length of the spatial domain divided by the wavelength $\boxed{k=\frac{2\pi a\cos\varphi}{\lambda}}$

$v\left[\frac{\text{m}}{\text{s}}\right]=\lambda[\text{m}]\times f\left[\frac{\text{Hz}}{\text{s}}\right]$ Wave speed $=$ wavelength $\times$ frequency

$\Phi(x,y,t)=\Phi_0+\Phi'\sin[k(x-ct)]\cos(ly)$ | Φ' = amplitude, k = wave # in x direction, l = wave # in y direction; c = phase speed;

$\boxed{c=\frac{v}{k}}$, where c = phase speed, $v\{\text{nu}\}$ = frequency, k = wavenumber

CIRCULATION THEOREM $\quad C=C(p,T,\phi,A)$

$\boxed{C=\oint_\Gamma \vec{V}\bullet d\vec{r}}\left[\frac{\text{m}^2}{\text{s}}\right]$ | $\leftarrow$ Definition of Circulation. It is a macroscopic measure of rotation for a finite area of the fluid. $\Rightarrow \boxed{C=\oint_\Gamma u\,dx+\oint_\Gamma v\,dy+\oint_\Gamma w\,dz}$

Sign convention: Always $\int$ CCW around Γ : + result $\Rightarrow$ CCW net flow; & vice versa

$\boxed{\frac{DC_a}{Dt}=\frac{D}{Dt}\oint \vec{U}_a\bullet d\vec{l}=-\oint\frac{1}{\rho}dp}\left[\frac{\text{m}^2}{\text{s}^2}\right]$ **Circulation Theorem. (Absolute Circulation)** RHS is solenoidal term (ST)

For a barotropic fluid, the density is a function only of pressure, & the ST is zero, $\therefore$, the absolute circulation is conserved following the motion.

In a baroclinic fluid, circulation may be generated by the ST (ex: sea-breeze circulation)

$C_e=2\Omega\langle\sin\phi\rangle A=2\Omega A_e$ | Circulation in the horizontal plane due to the rotation of the $A_e=A\sin\bar{\phi}$ = projection of area A in the equatorial plane

$C=C_a-C_e=C_a-2\Omega A_e$ | It is more convenient to work w/ relative circ, C, as opposed to absolute, C_a, since a portion of abs (C_e) is due to E's rotation

$\boxed{\frac{DC}{Dt}=-\oint_\Gamma\frac{dp}{\rho}-2\Omega\frac{DA_e}{Dt}}$ **Bjerknes circulation theorem**

$-\oint_\Gamma\frac{dp}{\rho}=$ Solenoidal term. $\quad -2\Omega\frac{DA_e}{Dt}=$ stretching term

For a *barotropic fluid*, the density is a function only of pressure, so the solenoidal term = 0

Stretching term change w/ time: $\int_{C_1}^{C_2}DC=-2\Omega\int_{A_{e1}}^{A_{e2}}dA_e\Rightarrow\boxed{C_2-C_1=-2\Omega\left(A_2\sin\phi_2-A_1\sin\phi_1\right)}\left[\text{m}^2\ \text{s}^{-1}\right]$

$\therefore C$ changes if either A or ϕ changes. If $\Delta A=0$ & parcel moves north $\Rightarrow C\downarrow$

If parcel stays at same latitude: $\boxed{C_2-C_1=-2\Omega\sin\phi\left(A_2-A_1\right)}$ If $A\uparrow$, then $C\downarrow$

If $A\downarrow$, then $C\uparrow$, analagous to angular momentum conservation & changes.

Mean tangential velocity : $V=(C/\text{circumference})[\text{m/s}]$

CIRCULATION & VORTICITY $\quad \Delta C=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\Delta x\Delta y\Rightarrow\boxed{\Delta C=\zeta\cdot\Delta A}$

$\boxed{C=-bhl}$ where vorticity = -b

Whenever the circulation $\neq 0$, there must be a non-zero *net* vorticity within Γ. Similarly, when $C=0$, there can be no *net* vorticity inside the curve.

$$\delta C=u\delta x+\left(v+\frac{\partial v}{\partial x}\delta x\right)\delta y-\left(u+\frac{\partial u}{\partial y}\delta y\right)\delta x-v\delta y\Rightarrow\boxed{\delta C=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\delta x\delta x}$$

$\boxed{\frac{\delta C}{\delta A}=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\equiv\zeta}$ | $\therefore$ for a finite area, C divided by A gives the average normal component of vorticity in the region. Vorticity of a fluid in solid-body rotation is $2\times$ the angular velocity of rotation. Vorticity may $\therefore$ regarded as a measure of the local angulare velocity of the field.

VORTICITY - The microscopic measure of rotation in a fluid, is a vector field defined as the curl of vorticity. Absolute: $\vec{\omega}_a\equiv\vec{\nabla}\times\vec{U}_a$ Relative: $\vec{\omega}\equiv\vec{\nabla}\times\vec{U}$ | $\vec{\omega}=\vec{q}$

$\therefore\ \vec{\omega}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x},\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$ | Vertical components of abs. & rel vort are: $\boxed{\eta\equiv\hat{k}\bullet\left(\vec{\nabla}\times\vec{U}_a\right)}$ $\quad\boxed{\zeta\equiv\hat{k}\bullet\left(\vec{\nabla}\times\vec{U}\right)}\left[\text{s}^{-1}\right]$

$\zeta>0\Rightarrow$ cyclonic (CCW) motion/storms in NH. $\zeta<0\Rightarrow$ cyclonic storms in SH.

$\therefore$ Distribution of ζ is an excellent diagnostic for weather analysis. η tends to be conserved following the motion at mid-trop levels $\Rightarrow$ this is basis for forcast model w/ BVE

$\hat{k}\bullet\vec{\nabla}\times\vec{U}_e=2\Omega\sin\phi\equiv f$

$f=2\Omega\sin\phi=2\vec{\Omega}\bullet\hat{k}$

$f(\text{NH})>0;\ f(\text{SH})<0$

Planetary vorticity is the local vertical component of the vorticity of the Earth due to rotation, = the *coriolis parameter,* the component of the planetary vorticity $2\vec{\Omega}$ along the local vertical.

$\therefore\ \boxed{\eta=\zeta+f},\ \boxed{\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}},\ \eta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f$

$$\boxed{\frac{D(\zeta+f)}{Dt}=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\rho}{\partial y}\frac{\partial p}{\partial x}\right)}$$

diverg/stretch, tilting, solenoidal

$\boxed{f(y)\Rightarrow v\frac{df}{dy}=\frac{Df}{Dt}}$ Coriolis parameter depends only on y. .

$\boxed{\frac{D(\zeta+f)}{Dt}=-f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$ $\quad\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=$ Horizontal diverg

BAROTROPIC VORTICITY EQUATION

The vertical component of absolute vort is conserved following horizontal motion

$\frac{d_H(\zeta+f)}{dt}=\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}+v\frac{\partial f}{\partial y}=0$ | Local relative vort changes are caused by absolute vorticity advection only

For mid-latitude β-plane the eqn has the form: $\boxed{\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\zeta+\beta v=0}$

$\boxed{\frac{\partial\zeta}{\partial t}=-u\frac{\partial\zeta}{\partial x}-v\frac{\partial\zeta}{\partial y}-v\frac{\partial f}{\partial y}};\quad\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\equiv\vec{\nabla}^2\psi$

$\boxed{\frac{\partial}{\partial t}\vec{\nabla}^2\psi=-u_\psi\frac{\partial}{\partial x}\vec{\nabla}^2\psi-v_\psi\frac{\partial}{\partial y}(\vec{\nabla}^2\psi+f)}$ | $u_\psi=-\frac{\partial\psi}{\partial y},v_\psi=\frac{\partial\psi}{\partial x}$

$\boxed{\frac{d_H(\zeta+f)}{dt}=-u\frac{\partial(\zeta+f)}{\partial x}-v\frac{\partial(\zeta+f)}{\partial y}}$

$\frac{\partial\zeta}{\partial t}=-u\frac{\partial(\zeta)}{\partial x}-v\frac{\partial(\zeta)}{\partial y}-v\beta\Rightarrow\boxed{\frac{\partial\zeta}{\partial t}\approx -v\beta}$ where $\boxed{\beta=\frac{\partial f}{\partial y}=\frac{2\Omega}{a}\cos\phi}$

$v<0 \qquad v>0$

CCW

$v\beta<0 \qquad v\beta>0$

$\therefore\ \frac{\partial\zeta}{\partial t}>0$ on LHS of vortex; RHS $\frac{\partial\zeta}{\partial t}<0$

$\boxed{\frac{d_H(\zeta)}{dt}=-u\frac{\partial(\zeta)}{\partial x}-v\frac{\partial(\zeta)}{\partial y}-v\beta}$

Internal gravity waves

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}+g=0$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial\theta}{\partial t}+u\frac{\partial\theta}{\partial x}+w\frac{\partial\theta}{\partial z}=0,$$

where $\theta=\frac{p}{\rho R}\left(\frac{p_s}{p}\right)^\kappa\Rightarrow\ \ln\theta=\gamma^{-1}\ln p-\ln\rho+\text{constant}$

Rossby Number : One way to estimate the importance of the earth's rotations on fluid motions is to calculate the RN, which is simply the ratio of the wind accel to the Coriolis acceleration: $\frac{U^2/L}{f_cU}=\frac{U}{f_cL}=Ro$. For typical synoptic scale motions it turns out that the wind accel is small (~10% or less) compared to the Coriolis, & the wind is nearly geostrophic. Thus, for geostrophic balance to be valid, the Ro must be small (~0.1), or the accel of the winds are ~ an order of magnitude smaller than the coriolis accel. In the tropics however, this is not the case, and Ro is in the order of 1, which means that the flow accel & the coriolis accel are about the same.

Horizontal Momentum Equations in log pressure coordinates with curvature terms

Since there is a single monotonic relationship between pressure & height, we can use p as the independent vertical coordinate and height (geopotential) as a dependent variable. The thermodynamic state of the atmosphere is then specified by the fields of $\Phi(x,y,p,t)$ and $T(x,y,p,t)$

$\frac{D\vec{V}}{Dt}+f\hat{k}\times\vec{V}=-\vec{\nabla}\Phi$, $\quad f\hat{k}\times\vec{V}=f\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\0&0&1\\u&v&w\end{vmatrix}=f\left[\hat{i}(0w-v)-\hat{j}(0w-u)+\hat{k}(0v-0u)\right]$

$\Rightarrow\boxed{fu\hat{j}-fv\hat{i}}$

$-\vec{\nabla}\Phi=-\left[\frac{\partial\Phi}{\partial x}+\frac{\partial\Phi}{\partial y}\right]$

$\frac{Du}{Dt}-fv+\frac{\partial\Phi}{\partial x}=F_x\Rightarrow\boxed{\frac{Du}{Dt}-\left(f+\frac{u\tan\phi}{a}\right)v+\frac{\Phi_\lambda}{a\cos\phi}=F_x}$

$\frac{Dv}{Dt}+fu+\frac{\partial\Phi}{\partial y}=F_y\Rightarrow\boxed{\frac{Dv}{Dt}+\left(f+\frac{u\tan\phi}{a}\right)u+\frac{\Phi_\phi}{a}=F_y}$

Sign conventions for vorticity, coriolis, & cyclonic flow

North Hem	**South. Hem**
$f > 0$	$f < 0$
$\zeta > 0$	$\zeta > 0$
CCW around Low	CCW around High
Cyclonic	Anti-cyclonic
$\left.\frac{\partial \bar{T}}{\partial y}\right\|_p < 0$	$\left.\frac{\partial \bar{T}}{\partial y}\right\|_p$
$U_T > 0$	U_T

$\zeta < 0$	$\zeta < 0$
CW around High	CW around Low
Anti-Cyclonic	Cyclonic

Ridge ∩	Trough ∩
Trough ∪	Ridge ∪

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In both hemispheres:

- Cyclonic = rotating in the same sense as the Earth's rot. Ω
- Coriolis & Cyclonic always have the same sign
- In westerly flow, a trough (& enhanced cyclogenesis) develops on the leeward side of the mountain range.
- Growing baroclinic wave tilts westward w/ height

Note: $U_T \propto -\frac{1}{f}\frac{\partial \bar{T}}{\partial y}$

Vorticity $\boxed{\zeta = \frac{V}{R_s} - \frac{\partial V}{\partial n}}$ $\boxed{\zeta_g \equiv \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}}$

Potential Vorticity

$P \equiv (\zeta_\theta + f)\left(-g\frac{\partial \theta}{\partial p}\right) = \text{Const.}\ \left[\text{K kg}^{-1}\ \text{m}^2\ \text{s}^{-1}\right] \leftarrow$ Ertel's PV

$1\ \text{PVU} = 10^{-6}\ \text{K kg}^{-1}\ \text{m}^2\ \text{s}^{-1}$ | PVU < 2 $\Rightarrow$ troposphere; PVU > 2 $\Rightarrow$ stratosphere

In a homogenous incompressible fluid:

$\frac{\zeta + f}{h} = \frac{\eta}{h} = \text{Barotropic PV} = \text{Constant}$ $\boxed{\frac{\zeta_1 + f_1}{h_1} = \frac{\zeta_2 + f_2}{h_2}}$

1) ζ_g 2) $\bar{\nabla}\zeta_g$ 3) $-\bar{V}_g \bullet \bar{\nabla}\zeta_g$ 4) $-v_g\frac{df}{dy}$

1)	< 0		> 0		< 0
2)	0	> 0	0	< 0	0
3)	0	< 0	0	> 0	0
4)	0	> 0	0	< 0	0

Horiz gradient of vort is zero, does nothing to ↑ or ↓ vort, it can not intensify the wave. To intensify wave, need to have gradient of vort change

NorthSouth gradient of planetary vort is + in both hemispheres.

Useful equations $\boxed{dy = a\,d\phi}$

$\boxed{\beta\left[\text{s}^{-1}\text{m}^{-1}\right] \equiv \frac{df}{dy} = \frac{d2\Omega\sin\phi}{a\,d\phi} = \frac{2\Omega\cos\phi}{a}}$, $\boxed{u_g = -\frac{1}{f}\frac{\partial \phi}{\partial y}}$ $\boxed{v_g = \frac{1}{f}\frac{\partial \phi}{\partial x}}$

Dynamical Eqns in Pressure Coordinates, neglecting curvature

$\frac{d\boldsymbol{V}}{dt} = -\boldsymbol{\nabla}\Phi - 2\boldsymbol{\Omega}\times\boldsymbol{V}$

$\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}$

$p = \rho R T$

$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \varpi}{\partial p} = 0$

$c_p\frac{dT}{dt} - \alpha\varpi = \frac{dq}{dt}$

where: $\left(\frac{d}{dt}\right)_p = \left(\frac{\partial}{\partial t}\right)_p + u\left(\frac{\partial}{\partial x}\right)_p + v\left(\frac{\partial}{\partial y}\right)_p + \varpi\left(\frac{\partial}{\partial p}\right)_p$

and $\varpi = \frac{dp}{dt}$

$\boxed{\text{Diverg}_g \equiv \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}}$

$\partial\omega = -\bar{\nabla}\bar{V}\partial p \Rightarrow \boxed{\omega(p) = -\int_{p_0}^{p}\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)dp}$

Quasi - Geostrophic Approximation

$\boxed{\frac{d_g\boldsymbol{V_g}}{dt} = -f_0\boldsymbol{k}\times\boldsymbol{V_a} - \beta y\boldsymbol{k}\times\boldsymbol{V_g}}$, where:

$\left(\frac{d_g}{dt}\right)_p = \left(\frac{\partial}{\partial t}\right)_p + u_g\left(\frac{\partial}{\partial x}\right)_p + v_g\left(\frac{\partial}{\partial y}\right)_p$

$\boxed{\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)_p + \frac{\partial \varpi}{\partial p} = 0}$, $\boxed{\boldsymbol{V_g} \equiv \frac{1}{f_0}\boldsymbol{k}\times\nabla\Phi}$

$\boxed{\left(\frac{\partial}{\partial t} + V_g\cdot\nabla\right)\left(-\frac{\partial \Phi}{\partial p}\right) - \sigma\varpi = \frac{\kappa J}{p}}$, where: $\kappa \equiv R/c_p$

$\therefore$ 4 eqns & 4 unknowns: $\Phi, V_g, V_a, \& \varpi$

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Quasi - Geostophic Vorticity Equation

$\frac{d_g\zeta_g}{dt} = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta v_g$

$\boxed{\frac{\partial \zeta_g}{\partial t} = -\bar{\boldsymbol{V}}_g\cdot\bar{\boldsymbol{\nabla}}(\zeta_g + f_0) + f_0\frac{\partial \varpi}{\partial p}}$, $\left[\text{s}^{-1}\right]$

where: $f_0\frac{\partial \varpi}{\partial p} =$ divergence term

$-V_g\cdot\nabla(\zeta_g + f_0) =$ | Advection of abs. vort. by the geostophic wind

$\boxed{\frac{\partial \zeta_g}{\partial t} = -\bar{\boldsymbol{V}}_g\cdot\bar{\boldsymbol{\nabla}}\zeta_g - \beta v_g + f_0\frac{\partial \varpi}{\partial p}}$, $\left[\text{s}^{-1}\right]$, where:

$-\bar{\boldsymbol{V}}_g\cdot\bar{\boldsymbol{\nabla}}\zeta_g = -u_g\frac{\partial \zeta_g}{\partial x} - -v_g\frac{\partial \zeta_g}{\partial y}$ = Adv. of geo. rel. vorticity

βv_g = advection of planetary vorticity

Short Waves (L<3000 km): Adv. of ζ_g dominates over adv. of planetary vort $\Rightarrow$ wave moves east

Long Waves (L>10000 km): Advection of planetary vort dominates over adv. of ζ_g $\Rightarrow$ wave moves westward

Quasi - Geostrophic Geopotential Tendency Equation (QGGTE)

$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right]\chi = -f_0\boldsymbol{V_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) + \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\nabla\left(\frac{\partial \Phi}{\partial p}\right)\right]$

(A) $\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2} =$ | Local geopotential tendency which is proportional to $-\chi$; $\boxed{\chi \equiv \frac{\partial \Phi}{\partial t}}$

(B) $-f_0\boldsymbol{V_g}\cdot\nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right) =$ | Distribution of vort. advection $\propto$ to advection of abs. vorticity; Dominant forcing term in upper trop

(C) $\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\boldsymbol{V_g}\cdot\nabla\left(\frac{\partial \Phi}{\partial p}\right)\right] =$ | Differential temp. advection (DTA); DTA $\propto \chi$. DTA AKA thickness advection.

(C) is the major mechanism for the amplification or decay of mid-lat synoptic scale systems. This term MUST be nonzero in order for a midlatitude synoptic-scale baroclinic wave to intensify through baroclinic processes.

(C) involves the rate of change with pressure of the horizontal thickness advection. The thickness advection tends to be strongest in the lower troposphere below the 500mb trough's and ridgelines in a developing baroclinic wave.(above 500 mb temperature gradients are weaker and geopotential and temperature isolines become more parallel) . In contrast to term B, the forcing of term C is concentrated in the lower troposphere. Term C will deepen upper level troughs and build upper level ridges in developing waves.

$\therefore\ \chi > 0 \Rightarrow$ warm advection $\Rightarrow \Phi\uparrow$ w/ time

$\chi < 0 \Rightarrow$ cold advection $\Rightarrow \Phi\downarrow$ w/ time

If the distribution of Φ is known at a given time, then terms B and C may be regarded as known forcing(intensification) functions , and the GTE is a linear partial diff eq in the unknown χ.

Quasi - Geostrophic Analysis w / the QGGTE

- Geostrophic advection of absolute vorticity: 1) Short waves move eastward
2) Upper-level vorticity advection does not affect the strength of midlatitude synoptic_scale baroclinic waves
- Differential thickness advection: 1) Horizontal temperature advection must be nonzero in order that a midlatitude synoptic-scale baroclinic wave intensify through baroclinic processes

Quasi - geostrophic approximation

for synoptic scale motions, the twin requirements of hydrostatic and geostrophic balance constrain the baroclinic motions so that to a good approximation, the structure and evolution of the 3-D flow field is determined approximately by the isobaric distribution of the geopotential $[\theta(x,y,p,t)]$

Requirements: 1) Geostrophic or almost(quasi) Geostrophic balance
2) Hydrostatic balance 3) Mid-latitude synoptic scale motions/systems
4) Strong baroclinicity 5) isobaric coordinate system

QGVE states that the local rate of change of geostrophic vorticity is given by the sum of the advection of the absolute vorticity by the geostrophic wind plus the concentration or dilution of vorticity by stretching or shrinking of fluid columns(the divergence term/effect). It is useful because if the evolution of vorticity can be predicted, then the evolution of the geopotential field can be predicted along with the geostrophic winds and temp distributions. Vorticity advection will only move the wave pattern, it will not strengthen the disturbance.

POTENTIAL VORTICITY (PV) is always in some sense a measure of the ratio of the absolute vorticity, η, to the effective depth of the vortex.

Where the effective depth is just the differential difference between potential temp. surfaces measured in pressure units: ($-\partial\theta/\partial P$). Or a simplified version is if you assume a homogenous incompressible fluid, the horizontal depth must be inversely proportional to the depth, h, of the fluid parcel. This yields: $(\zeta + f)/h = \eta/h = \text{const.}$

Ex: A trough always develops on the leeward side of a mtn. in both hemispheres
~~~~~~

$\frac{\partial\zeta}{\partial t} = -u\frac{\partial\zeta}{\partial x} - v\frac{\partial\zeta}{\partial y} - v\beta,$ | Barotropic Vorticity Eqn, where $\zeta = \nabla^2\psi$, $u = -\partial\psi/\partial y,\ v = \partial\psi/\partial x$

if $\psi = -\bar{u}y + (\bar{u}/k)\sin(kx)$

PERTURBATION METHOD

- Useful for the qualitative analysis of atmospheric waves, eg. the stability of a given BS flow w/ respect to small superposed perturbations

All field variables are divided into 2 parts:

1) *Basic state* portion (BS): assumed to be independent of t & longitude
2) *Perturbation* portion: local deviation of the field from the basic state

eg. 1: To create zonal avg: $u(x,y,t) \Rightarrow \bar{u}(y) + u'(x,y,t)$, where $\bar{u}(y)$ = basic state; $u'(x,y,t)$ = perturbations from the zonal mean

eg. 2: Complete zonal vel. field: $u(x,t) = \bar{u} + u'(x,t)$ || $\bar{u}$ = time & longitude-avrgd zonal vel. u' = deviation from that average

Then, inertial acceleration is:

$u\frac{\partial u}{\partial x} \Rightarrow \bar{u}\cancel{\frac{\partial \bar{u}}{\partial x}} + \boxed{\bar{u}\frac{\partial u'}{\partial x}} + u'\cancel{\frac{\partial \bar{u}}{\partial x}} + \cancel{u'\frac{\partial u'}{\partial x}}$ | $\bar{u}$ is constant so deriv = 0; 4th term can be neglected

Basic assumptions of perturbation theory:

1) Each dependent variable can be represented as the sum of some avgerage state (basic state) and a deviation from that state (perturbation)
2) Both the total field (BS + pert) & the BS fields satisfy the governing eqns
3) Perts are sufficiently small that all terms w/ products of pert quantities can be neglected

Then, the non-linear governing eqns are reduced to linear diff. eqns in the perturbation variables in which the BS variables are specified coefficients. Solutions of perturb eqns then determine characteristics such as: propogation speed, vertical structure, & conditions for growth & decay of the waves.

DISPERSION & GROUP VELOCITY

Dispersive Waves: (Rossby & gravity waves are dispersive)

- Phase speed of the waves change with their wavelength
- $k = k(c)$
- Speed of wave group is different from the avg phase speed of the indv Fourier components
- Shape of a wave group is not constant as the group propagates. The group generally broadens in time, ie. the energy is dispersed.
- For propagating waves, ν (frequency) depends on the wave # of the pert. as well as the physical properties of the medium. Thus, b/c $c = \nu/k$, the phase speed depends on k, (except in special case where $\nu \propto k$)
  $\therefore$ For waves in which c varies with k, the various sinusidal components get dispersed in time.
- In synoptic-scale Atm disturbances, the group velocity > phase velocity.
- $\boxed{C_{gx} = \partial\nu/\partial k}$ Group velocity

Non-Dispersive Waves:

Shallow water gravity waves

$\boxed{c = \bar{u} \pm \sqrt{gH}}$ (SW wave speed), valid only for waves where $\lambda \gg H$

SHALLOW WATER MODEL & EQNS: (u & v momentum, continuity)

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial h'}{\partial x};\quad \frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial h'}{\partial y};\quad \frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$

$$\frac{\partial(v\ \text{eqn})}{\partial x} - \frac{\partial(u\ \text{eqn})}{\partial y} \Rightarrow \frac{\partial\zeta'}{\partial t} + f_0\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0$$ | h'(Holton) $= \eta$ (Gill)

Solve for diverg in continuity & subst into above = linearized potential vorticity conservation law | $\Rightarrow \boxed{\frac{\partial\zeta'}{\partial t} - \frac{f_0}{H}\frac{\partial h'}{\partial t} = 0}$

Adjustment to balance: non-rotating fluid under the effect of gravity

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) = 0,$$ | Eqn in 1 variable only, h', solutions are 2-d shallow water gravity waves

- Steady state solution is rest with a flat free surface
- Adjustment is accomplished by shallow water gravity waves
- All initial energy is lost

Adjustment to balance: rotating fluid under the effect of gravity non-zero f_0;

$$\frac{\partial^2 h'}{\partial t^2} - c^2\left(\frac{\partial^2 h'}{\partial x^2} + \frac{\partial^2 h'}{\partial y^2}\right) + f_0 H\zeta' = 0$$ | The h' and ζ' fields are coupled

Assumptions: Horizontal scale is large compared w/ depth, so that hydrostatic approximation can be made. 1/3 of the PE released goes into the steady geostrophic flow. The remaining 2/3 is radiated away by inertia-gravity waves! The Equil. state depends on the initial state: the connection is conservation of PV.
Note: solution could not be derived merely by setting $\partial/\partial t = 0$ in SW eqns That would yield geostrophic balance and any distribution of h' would satisfy SW eqns. Only by combining SW eqns to obtain PV eqn, and requiring the flow to satisfy PV conservation at all intermediate times, can the degeneracy of the geostrophic final state be eliminated.

- $\partial v'/\partial x - \partial u'/\partial y \Rightarrow (5)$; subst ζ' into (5); solve for Div in(3)& subst..

Rossby Adj Problem: an anti-cyclone is produced where the fluid height is elevated,where the fluid height is depressed cyclonic rotation is observed.

ROSSBY WAVES (RW)

t_0 : $\zeta_0 = 0$. $t_1 : (\zeta + f)_{t1} = f_{t0}$ or $\zeta_{t1} = f_{t0} - f_{t1} = -\beta\delta y$
$\therefore\ \delta y > 0 \Rightarrow \zeta_{t1} < 0;\ \delta y < 0 \Rightarrow \zeta_{t1} > 0\ \therefore$ westward displacement of the pattern of vort max & mins due to advection by the induced velocity.
The meridional gradient of η resists meridional displacement & provides the restoring mechanism for RW. $\boxed{c = -\beta/k^2}$

Free Barotropic RW (BRW)

Dispersion relationship for BRW may be derived by finding wave-type solutions of the linearized BVE $\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\zeta + \beta v = 0$

$u = \bar{u} + u'$, $v = v'$, $\zeta = \partial v'/\partial x - \partial u'/\partial y = \zeta'$. Define ψ' according to:
$u' = -\partial\psi'/dy$, $v' = -\partial\psi'/dx$, from which we see that $\zeta' = \overline{\nabla}^2\psi'$

$\therefore$ perturbation form of BVE is $\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\overline{\nabla}^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0$

We seek solution of the form: $\psi' = \text{Re}\left[\Psi\exp(i\phi)\right]$, where $\phi = kx + ly - \nu t$
Subst ψ' into pert BVE gives: $(-\nu + k\bar{u})(-k^2 - l^2) + k\beta = 0 \ \Rightarrow$
$\nu = \bar{u}k - \beta k/(k^2 + l^2)$; since $c = \nu/k$, $\boxed{c - \bar{u} = -\beta/(k^2 + l^2)}$
where: ν = frequency; k & l are zonal & meridional wave #'s respectively.
RW: • Propagate westward w/ respect to the mean flow • Are dispersive
• Their phase speed increases rapidly with increasing wavelength
• Typical mid-lat synoptic-scale disturbance, where $l \approx k$, & zonal $\lambda \approx 6000$ km, RW speed rel to zonal flow ≈ -8 m/s

Rossby Radius of Deformation

$\lambda_R \equiv \frac{\sqrt{gH}}{f_0}$, | Horizontal length scale over which the height field adjusts during the approach to geostrophic equilibrium.

- When scale of motion < λ_R: adjustment ≈ non-rotating system adjustment
- When the scale of motion > λ_R, Coriolis is important $\Rightarrow$ geostrophic adj.

Baroclinic wave disturbances arise from a hydrodynamic instability of the midlatitude jet: Flow is hydrodyn.-ly unstable if 'a small disturbance introduced into it grows spontaneously drawing energy from the mean flow".

Barotropic Instability:

- associated with the horizontal shear in a jet-like current
- Waves grow by extracting kinetic energy from the mean flow
- African Easterly waves.

Baroclinic Instability:

- Associated with the vertical shear of a jet-like current
- Waves grow by extracting potential energy from the mean flow
- Midlatitude baroclinic waves

Normal Mode Instability Analysis Method

Linear analysis: 1) introduce a single wave mode of the form $\exp[ik(x - ct)]$
2) determine the conditions for which the phase speed, c, has an imaginary part, which is the condition for that mode to grow

$$\boxed{c = U_m - \frac{\beta(k^2 + \lambda^2)}{k^2(k^2 + 2\lambda^2)} \pm \delta^{1/2}},\quad \delta \equiv \frac{\beta^2\lambda^4}{k^4(k^2 + 2\lambda^2)^2} - \frac{U_T^2(2\lambda^2 - k^2)}{(k^2 + 2\lambda^2)}$$

$\lambda^2 \equiv \frac{f_0^2}{[\sigma(\delta p)^2]}, \sigma \equiv -\frac{RT_0}{p}\frac{d\ln\theta}{dp}$, | U_T tells shear strength; σ = Atm stability;

Variables that determine thee sign of σ: (β, U_T, K, λ)
$\sigma < 0$: an imaginary c, unstable mode; $\sigma = 0$: marginally stable.
$\sigma > 0$: Stable (or neutral, or non-amplifying) waves occur
Typical mid-lat values: $\sigma = 5°/100$mb, $U_T = 5.5$ m/s, $L_{\min} = 4340$ km
This 'simple' analysis therefore indicates that baroclinic instability is a primary mechanism for synoptic-scale wave development in the midlatitudes

Planetary Boundary Layer (PBL)

Stable Boundary Layer (SBL)

PBL & SBL Equations:

$$\frac{\partial A}{\partial t} = -\nabla \bullet \boldsymbol{V}A + Q_A$$

$$\frac{\partial A}{\partial t} = -\nabla \bullet \overline{\boldsymbol{V}}\overline{A} - \overline{\boldsymbol{V}'A'} + Q_{\overline{A}}$$

$$\overline{\boldsymbol{V}'A'} = -K_A \nabla \overline{A}$$

$$\frac{\partial A}{\partial t} = -\overline{\nabla} \bullet \overline{V}A + Q_A$$

$$Q_v = -\rho_a K_v \frac{\partial \overline{q}}{\partial z}$$

$$\boldsymbol{V}(0) = 0; \quad \boldsymbol{V}(H) \equiv \boldsymbol{V}_g$$

$$\frac{\tau}{\rho_a} = -\overline{\boldsymbol{V}'\boldsymbol{V}'} \neq f(z) \text{ in SBL}$$

$$\frac{\tau}{\rho_a} = -\overline{\boldsymbol{w}'\boldsymbol{U}'} = f(\ )',$$

where $\overline{U} = \sqrt{\overline{u}^2 + \overline{v}^2}$

$$U' = \ell' \frac{\partial \overline{U}}{\partial z} = -w'$$

$$\tau = \rho_a \ell^2 \left(\frac{\partial \overline{U}}{\partial z}\right)^2,$$

$$\ell = k_0 (z + z_0); \quad \ell \equiv \sqrt{\overline{(\ell')^2}}$$

$$u_* = \sqrt{\tau / \rho_a}$$

$$\frac{\partial \overline{U}}{\partial z} = \frac{u_*}{\ell} = \frac{u_*}{k_0 (z + z_0)}$$

$$\frac{\tau}{\rho} = K_m \frac{\partial \overline{U}}{\partial z},$$

$$K_m = \ell^2 \frac{\partial \overline{U}}{\partial z}$$

$$K_m = u_*^2 \Big/ \frac{\partial \overline{U}}{\partial z}$$

$$Q_m = -\rho_a u_*^2 = -\tau$$

1-D MODEL: ESTOQUE

SBL:

$$\frac{\partial Q_a}{\partial z} = 0,$$

$$r = \frac{K_H}{K_M} = \textbf{Inv. Pr. \# =1}$$

$$Q_A = -\textbf{const} \cdot \rho \cdot u_* A_*$$

$$Q_A = -\textbf{const} \cdot \rho \cdot K_A \frac{\partial \overline{A}}{\partial z}$$

$$\frac{\partial \overline{A}}{\partial z} = \frac{A_*}{k_0 z} \phi_A^*$$

$$\phi_h^* = \phi_m^* = \phi_q^* = \phi^*(Ri)$$

$$\phi^* = \left(1 + \alpha (Ri)\right)^{-1}, \quad \left| \begin{array}{l} \text{Forced} \\ \alpha = -3 \end{array} \right.$$

$$\phi^* = f(\partial\theta/\partial z) \neq f(\partial U/\partial z)$$

$$K(h) = \frac{u_*^2}{(\partial U/\partial z)_h}$$

$$\left(K \frac{\partial A}{\partial z}\right)_h = A_* u_*$$

3-D MODEL

$$\frac{\boldsymbol{F}}{m} = \boldsymbol{A} \Rightarrow \frac{\boldsymbol{F}}{m} = \frac{\partial \boldsymbol{V}}{\partial t}$$

$$\frac{\partial \boldsymbol{V}}{\partial t} = -\boldsymbol{V} \bullet \nabla \boldsymbol{V} - \alpha \nabla p - 2\Omega \times \boldsymbol{V} + \boldsymbol{g_a} - \Omega \times (\Omega \times \boldsymbol{R}) + \upsilon \nabla^2 \boldsymbol{V}$$

$$\boldsymbol{g}(z, \phi) \equiv \boldsymbol{g_a} + \boldsymbol{C}_e = -g\hat{\boldsymbol{k}} \approx 9.8 \text{ m s}^{-2}$$

$$\therefore \frac{\partial \boldsymbol{V}}{\partial t} = -\boldsymbol{V} \bullet \nabla \boldsymbol{V} - \alpha \nabla p - 2\Omega \times \boldsymbol{V} - g\hat{\boldsymbol{k}} + \upsilon \nabla^2 \boldsymbol{V}$$

Continuity Eqns

1) Compressible

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \bullet \boldsymbol{V} \Rightarrow \frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{V} \bullet \nabla \rho\right) = -\nabla \bullet \boldsymbol{V}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\nabla \bullet \rho \boldsymbol{V}} \quad \Big| \text{RHS} = \text{mass convergence}$$

2) Homogenous: $\rho = \rho(t)$; $\rho \neq \rho(x, y, z)$

$$\therefore \frac{\partial \rho}{\partial t} = -\rho \nabla \bullet \boldsymbol{V}$$

3) Steady state: $\rho = \rho(x, y, z)$ only; $\rho \neq \rho(t)$;

$$\therefore \frac{\partial \rho}{\partial t} = 0, \rightarrow 0 = -\nabla \bullet \rho \boldsymbol{V}$$

4) Anelastic: $\rho = \rho(z)$ only;

$$\rho_a \nabla_{\mathrm{H}} \bullet \boldsymbol{V_H} + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial \rho}{\partial t} = 0$$

5) Incompressible/non-divergent flow

$$-\frac{1}{\rho}\frac{d\rho}{dt} = \nabla \bullet \boldsymbol{V} = 0 \quad \Rightarrow \text{zero vel. diverg.}$$

Order ε

$$\frac{\Delta \rho_0}{\rho_a} \equiv \varepsilon \ll 1$$

$$p_{dyn} = \frac{1}{2}\rho v^2$$

PBL Continuity Eqn

$$\nabla \bullet \boldsymbol{V} = -\frac{1}{\rho}\frac{d\rho}{dt}$$

$$\therefore \nabla \bullet \boldsymbol{V} = 0 \quad \left| \begin{array}{l} \text{Inst. flow is incompress} \\ \text{to Order } \varepsilon \end{array} \right.$$

$$\therefore \nabla \bullet \overline{\boldsymbol{V}} = 0 \quad \left| \begin{array}{l} \text{Mean flow is incompress} \\ \text{to Order } \varepsilon \end{array} \right.$$

Eqn of motion for mean flow

$$\frac{d\boldsymbol{V}}{dt} = -2\Omega \times \boldsymbol{V} - \alpha \nabla p - g\hat{\boldsymbol{k}} + \upsilon \nabla^2 \boldsymbol{V} \quad \Rightarrow$$

$$\frac{d\boldsymbol{V}}{dt} = -2\Omega \times \overline{\boldsymbol{V}} - \overline{\alpha \nabla p} - g\hat{\boldsymbol{k}} + \upsilon \nabla^2 \overline{\boldsymbol{V}} - \nabla \bullet \overline{\boldsymbol{V}'\boldsymbol{V}'}$$

PBL hydrostatic assumption

$$\frac{\partial \overline{w}}{\partial t} = -\overline{\boldsymbol{V}} \bullet \nabla \overline{w} - \overline{\alpha \frac{\partial p}{\partial z}} + \hat{f}\overline{u} - g + \upsilon \nabla^2 \overline{w} - \nabla \bullet \overline{\boldsymbol{w}'\boldsymbol{V}'}$$

$$\Rightarrow \boxed{0 = -\overline{\alpha \frac{\partial p}{\partial z}} - g} \quad \Big| \text{ After scale analysis}$$

NB: $\frac{\partial \overline{w}}{\partial z} = -\left[\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y}\right]$ | *i.e.* Horiz convergence leads to vert velocity.

Hydrostatic Eqn in PBL

$$\boxed{(\alpha_a + \alpha_0)\frac{\partial p_0}{\partial z} = -g} \text{ OR } \boxed{\frac{\partial p_0}{\partial z} = (\rho_a + \rho_0) g}$$

- Static variation

PBL Ideal Gas Law: $\rho = \dfrac{p}{RT}$

$$\frac{d\rho}{\rho} = \frac{dp}{p} - \frac{dT}{T}$$

To order ε

$$\frac{\rho_0 + \rho^*}{\rho_a} = \left(\frac{p_0 + p^*}{p_a}\right) - \left(\frac{T_0 + T^*}{T_a}\right)$$

Results:

$$\boxed{\frac{\rho_0}{\rho_a} = \frac{p_0}{p_a} - \frac{T_0}{T_a}} \quad \left| \begin{array}{l} \text{with } \overline{\boldsymbol{V}} = 0 \\ \text{Static state} \end{array} \right.$$

$$\boxed{\frac{\rho^*}{\rho_a} = \frac{p^*}{p_a} - \frac{T^*}{T_a}} \quad \left| \begin{array}{l} \text{Effect of } \overline{\boldsymbol{V}} \text{ after} \\ \text{subtracting static} \end{array} \right.$$

Misc. Eqns:

$$\frac{1}{a}\frac{da}{dt} = \frac{d \ln a}{dt}$$

Reynolds averaging

$$\overline{ab} = \overline{(\overline{a} + a')(\overline{b} + b')} = \overline{a}\overline{b} + \overline{a'b'}$$

Vector Derivatives

$$\nabla f \bullet \boldsymbol{A} = \nabla \bullet (f\boldsymbol{A}) - f(\nabla \bullet \boldsymbol{A})$$

Curvilinear Coordinates (Dutton)

$\hat{x}_i = f_i(x_1,x_2,x_3),\quad i=1,2,3;\qquad NB:\ \hat{\boldsymbol{x}} = \tilde{\boldsymbol{x}}$ in Pileke's system

$\hat{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \hat{\boldsymbol{x}}(\boldsymbol{x}) \Rightarrow$ vector form of above, where function $\boldsymbol{f}(\boldsymbol{x})$ prescribes 1 and only 1 value of $\hat{\boldsymbol{x}}$ for each value of $\boldsymbol{x}$ and is such that the 3 coordinates are independent of eachother.

- Cylindrical polar & spherical coordinates are orthogonal curvilinear coords but not cartesian. Orthogonal systems have distinct advantages in meteorology.

Invertibility condition: The condition for transformation of $\hat{x}_i = f_i(x_1,x_2,x_3)$ to be uniquely invertible is that the Jacobian determinant does not vanish for any $\boldsymbol{x}$.

$$\left|J_x^{\hat{x}}\right| = \left|\frac{\partial(\hat{x}_1,\hat{x}_2,\hat{x}_3)}{\partial(x_1,x_2,x_3)}\right|$$

This condition also ensures that the new coordinates are independent. If $\left|J_x^{\hat{x}}\right|$ doesn't vanish, then reciprocal =

$$\left|J_{\hat{x}}^{x}\right| = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(\hat{x}_1,\hat{x}_2,\hat{x}_3)}\right|$$

& it can be shown that $\left|J_x^{\hat{x}}\right| \bullet \left|J_{\hat{x}}^{x}\right| = 1$

Nonorthogonal Curvilinear Coordinates (Dutton)

When cooordinates fail to be orthogonal, we must use 2 sets of basis vectors & 2 sets of components in order to be able to determine components with scalar products. Moreover, there is no longer any advantage to having basis vectors of unit length, & instead the magnitudes of the basis vectors will carry the necessary information on distance scaling.

Summation Convention: requires sum on repeated indices when they appear on 2 quantities that are multiplied by eachother.

$e.g.:\ \vec{\boldsymbol{A}}\bullet\vec{\boldsymbol{B}} = \sum_{k=1}^{3} A_k B_k = A_k B_k \qquad e.g.:\ (\vec{\boldsymbol{A}}\bullet\vec{\boldsymbol{B}})(\vec{\boldsymbol{C}}\bullet\vec{\boldsymbol{D}}) = A_i B_i C_j D_j$

$e.g.$ If $\vec{\boldsymbol{A}} = \boldsymbol{i}_j A_j$ & $\vec{\boldsymbol{B}} = \boldsymbol{i}_k B_k$ then $\vec{\boldsymbol{A}}\bullet\vec{\boldsymbol{B}} = (\boldsymbol{i}_j\bullet\boldsymbol{i}_k) A_j B_k = \delta_{jk} A_j B_k = A_j B_j$

Denominator convention: superscript appearing in denominator = subscript

Covariant & Contravariant

Expanding $\tilde{\boldsymbol{x}}(\boldsymbol{x})$ w/ the chain rule yields the following two expansions:

$$d\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \tilde{x}^j} d\tilde{x}^j \qquad d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j = (\nabla \tilde{x}^i)\bullet d\boldsymbol{x}$$

& thus the two vectors $\vec{\tau}_j$ & $\vec{\eta}^i$ appear. (Pielke Fig. 6.1)

$$\vec{\tau}_j = \frac{\partial}{\partial \tilde{x}^j}\left(x^1\vec{\boldsymbol{i}} + x^2\vec{\boldsymbol{j}} + x^3\vec{\boldsymbol{k}}\right) \Rightarrow \boxed{\vec{\tau}_j = \frac{\partial \boldsymbol{x}}{\partial \tilde{x}^j}} \qquad N.B.,\ \vec{\tau}_j:$$

- $\vec{\tau}_j$ reveals the variation of the position vector as it traces out a curve in which $\tilde{x}^j$ varies & the other two coordinates are constant (acc. to partial deriv def)
- $\vec{\tau}_j$ is tangent to the curve along which only $\tilde{x}^j$ varies.
- if nonorthogonal coords, then tangent vector $\vec{\tau}_3$ to the curve on which $\tilde{x}^1$ & $\tilde{x}^2$ are constant does not have to coincide with the normal to the sfc $\tilde{x}^3 = \text{const}$
- $\vec{\tau}_3$ must be orthogonal to the vectors $\vec{\eta}^1$ & $\vec{\eta}^2$ that are normal to the sfcs on which $\tilde{x}^1$ & $\tilde{x}^2$ are constant

$$\vec{\eta}^i = \vec{\boldsymbol{i}}\frac{\partial \tilde{x}^i}{\partial x^1} + \vec{\boldsymbol{j}}\frac{\partial \tilde{x}^i}{\partial x^2} + \vec{\boldsymbol{k}}\frac{\partial \tilde{x}^i}{\partial x^3} = \vec{\nabla}\tilde{x}^i$$

$\vec{\eta}^i$ is normal to the sfc $\tilde{x}^i = \text{const}$.

NB: if orthogonal coords, these 2 sets of of vectors are identical direction-wise.

Covariant & Contravariant (Dutton)

Next step is to determine how basis vectors behave under further transformation. Let $\tilde{z}^i = \tilde{z}^i(x^1,x^2,x^3),\ i=1,2,3$ define another set of curvilinear coordinates

$\therefore$ the position vector differential becomes $\qquad d\boldsymbol{x} = \dfrac{\partial \boldsymbol{x}}{\partial \tilde{z}^i} d\tilde{z}^i = \tilde{\tau}_i d\tilde{z}^i$

while the coordinate differential becomes $\qquad d\tilde{z}^i = (\vec{\nabla}\tilde{z}^i)\bullet d\boldsymbol{x} = \tilde{\eta}^i \bullet d\boldsymbol{x}$

But, the coordinates $\hat{x}^i$ are also a function of $\boldsymbol{x}$ and the relation can be inverted to give $\boldsymbol{x} = \boldsymbol{x}(\hat{x})$. Thus, we may find the appropriate functions so that $\tilde{z}^i$ may be expressed as a transformation of the $\hat{\boldsymbol{x}}$ cooordinates in the form $\tilde{z}^i = \tilde{z}^i(\hat{x}^1,\hat{x}^2,\hat{x}^3)$

Now, apply chain rule to calculate that
$$\boxed{\tilde{\boldsymbol{\tau}}_i = \frac{\partial \boldsymbol{x}}{\partial \tilde{z}^i} = \frac{\partial \boldsymbol{x}}{\partial \hat{x}^k}\frac{\partial \hat{x}^k}{\partial \tilde{z}^i} = \frac{\partial \hat{x}^k}{\partial \tilde{z}^i}\hat{\boldsymbol{\tau}}_k} \qquad (32)$$

This relation shows that the tangent vectors have a specific law of transformation whose characterisitics are revealed by the placement of variables & indices in the derivative $\partial\hat{x}^k/\partial\tilde{z}^i$, the **position of variables controlling the differentiation** & the **position of indices controlling the summation**. Similarly:

Covariant & Contravariant (Dutton) cont..

$$\boxed{\tilde{\eta}^i = \vec{\nabla}\tilde{z}^i = \boldsymbol{i}_l\frac{\partial \tilde{z}^i}{\partial x^l} = \boldsymbol{i}_l\frac{\partial \tilde{z}^i}{\partial \hat{x}^k}\frac{\partial \hat{x}^k}{\partial x^l} = \frac{\partial \tilde{z}^i}{\partial \hat{x}^k}\hat{\boldsymbol{\eta}}^k} \qquad (33)$$

$\therefore$ the normal vectors also have there own law of transformation, whch is evidently different from (32) in both the differentiation & the summation.

It is customary to choose the law of transformation of the tangent vectors as the one to compare w/ other types, & hence quantities that transform like the tangent vectors are called ***covariant***. For a scaler ϕ, the $\partial\phi/\partial\tilde{z}^i$ after chain rule becomes:

$$\frac{\partial\phi}{\partial\tilde{z}^i} = \frac{\partial\phi}{\partial\hat{x}^k}\frac{\partial\hat{x}^k}{\partial\tilde{z}^i}$$

and so is covariant. In contrast we have (below):

$$\tilde{f}_i = \frac{\partial x^j}{\partial\tilde{x}^i} f_j \quad \leftarrow (\text{Pielke})$$

$$d\tilde{z}^i = \frac{\partial\tilde{z}^i}{\partial x^k}dx^k = \frac{\partial\tilde{z}^i}{\partial\hat{x}^j}\frac{\partial\hat{x}^j}{\partial x^k}dx^k = \frac{\partial\tilde{z}^i}{\partial\hat{x}^j}d\hat{x}^j$$

so that the coordinate differentials transform like the normal vectors. Such quantities are called ***contravariant*** to indicate that they behave like the normal vectors, not like the tangent vectors.

Covariant & Contravariant (Pielke)

Covariant : $\boxed{\tilde{f}_i = \dfrac{\partial x^j}{\partial\tilde{x}^i} f_j}$, where $\tilde{f}_i$ is a 1st order tensor

- $\tilde{f}_i \equiv$ covariant if $x^i \to \tilde{x}^i$ is given by above transformation
- Use of subscript denotes that $\tilde{f}_i$ is a covarient vector ($i.e.$ tensor of order 1)
- Superscript in the denominator of a derivative ($e.g.\ \partial/\partial x^j$) $\equiv$ covariant, by convention

Contravariant : $\boxed{\tilde{f}^i = \dfrac{\partial\tilde{x}^i}{\partial x^j} f^j}$

- $\tilde{f}^i$

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Kinetic Energy

$KE = \frac{1}{2}m\vec{V}^2 \rightarrow \frac{KE}{m} = \frac{1}{2}\vec{V}^2\ [\text{m}^2\ \text{s}^{-2}]$ | Dealing w/ fluid per unit mass

$\frac{MKE}{m} = \frac{1}{2}\overline{\vec{V}}^2$

$\frac{TKE}{m} = \bar{e} = \frac{1}{2}\overline{\vec{V}'^2} \Rightarrow \boxed{\bar{e} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)[\text{m}^2\ \text{s}^{-2}]}$ | Mean TKE

NB: $\vec{\tau}_j$ stress $[\text{N m}^{-2}]$; momentum $[\text{kg m s}^{-1}]$

Fluxes	Kinematic Flux	Tensor order
$\tilde{F}\{Q_M\}$ momentum $[\text{N m}^{-2}]$ or: $[(\text{kg m s}^{-1})(\text{m}^2\ \text{s}^{-1})]$	$F = \dfrac{\tilde{F}}{\rho_{air}}[\text{m}^2\ \text{s}^{-2}]$	2nd (9 comp)
$\tilde{Q}_H$ heat $[\text{J m}^{-2}\ \text{s}^{-1}]$	$Q_H = \dfrac{\tilde{Q}_H}{\rho_{air}C_{p_{air}}}[\text{K m s}^{-1}]$	1st (3 comp)

- Kinematic fluxes are in units that can be measure directly.
- Flux is the rate of transfer of a quantity across a unit area

TKE

$\text{Stress}_{\text{Re}}$ = TKE – strain

$$\overline{u_j'u_i'} = \frac{1}{3}\delta_{ij}\overline{u_k'u_k'} - k_{ijkl}\left[\frac{\partial\bar{u}_k}{\partial x_l} + \frac{\partial\bar{u}_l}{\partial x_k}\right]$$

$\bar{e} = \frac{1}{2}\overline{u_i'^2} \rightarrow |\ \bar{e} =$ summed velocity variances divided by 2

$\frac{D}{Dt}$
~~~~~~

VECTOR & TENSOR ANALYSIS

Temporally varying, differentiable cordinate transformations are the proper kinematic abstraction of both fluid motion & the motion of the atmosphere.

$$\nabla \equiv \hat{\boldsymbol{i}}\frac{\partial}{\partial x}+\hat{\boldsymbol{j}}\frac{\partial}{\partial y}+\hat{\boldsymbol{k}}\frac{\partial}{\partial z}$$

| Gradient Operator, del Operator

$$\nabla\phi = \hat{\boldsymbol{i}}\frac{\partial\phi}{\partial x}+\hat{\boldsymbol{j}}\frac{\partial\phi}{\partial y}+\hat{\boldsymbol{k}}\frac{\partial\phi}{\partial z}$$

Gradient of the scalar function $\phi(x,y,z,t)$

- $\nabla\phi$ points in the direction in which the field ϕ increases most rapidly
- ∇p = gradient of pressure
- $-\nabla p$ = pressure gradient
- $\nabla\phi$ is $\perp$ to lines of constant ϕ

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

Divergence of the vector function $\vec{A}$

- positive when fluid is expanding; neg when...
- The scalar $\nabla \bullet \vec{A}$ is called the divergence of the vector field $\vec{A}$ b/c it is a measure of the tendency of the field lines of $\vec{A}$ to diverge or converge

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

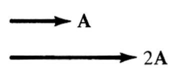
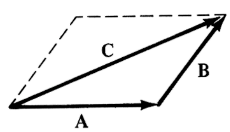
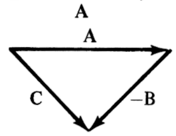
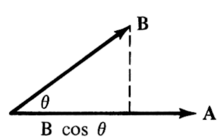
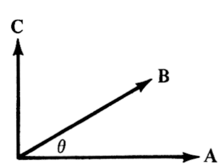
Curl of $\vec{A}$

- Curl is a measure of the tendency of a vector field to rotate at a point

Curvilinear coordinates

$\hat{x}_i = f_i(x_1,x_2,x_3), \quad i=1,2,3;$

$\hat{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) = \hat{\boldsymbol{x}}(\boldsymbol{x}) \Rightarrow$ vector form of above, where the function $\boldsymbol{f}(\boldsymbol{x})$ prescribes one and only one value of $\hat{\boldsymbol{x}}$ for each value of $\boldsymbol{x}$ and is such that the 3 coordinates are independent of eachother.

Operation		Definition and geometric significance		Analytic form	Properties
$\mathbf{C} = \alpha\mathbf{A}$	Multiplication by a scalar α	**C** is $\lvert\alpha\rvert$ times as long as **A** and in the same direction if $\alpha > 0$, the opposite direction if $\alpha < 0$.		$\mathbf{C} = \alpha\mathbf{A} = \alpha A_x\mathbf{i} + \alpha A_y\mathbf{j} + \alpha A_z\mathbf{k}$	$\alpha\mathbf{A} = \mathbf{A}\alpha$ $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$
$\mathbf{C} = \mathbf{A} + \mathbf{B}$	Addition	Move tail of **B** to tip of **A**; the sum is the vector from the tail of **A** to the tip of **B**.		$\mathbf{C} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}$	$\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$ $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
$\mathbf{C} = \mathbf{A} - \mathbf{B}$	Subtraction	Add **A** and **—B**.		$\mathbf{C} = (A_x - B_x)\mathbf{i} + (A_y - B_y)\mathbf{j} + (A_z - B_z)\mathbf{k}$	$\mathbf{A} - \mathbf{B} = -\mathbf{B} + \mathbf{A}$ $(\mathbf{A} - \mathbf{B}) + \mathbf{C} = (\mathbf{A} + \mathbf{C}) - \mathbf{B}$
$C = \mathbf{A} \cdot \mathbf{B}$	Scalar product (dot product)	$C = AB\cos\theta$, so the length of **B** projected onto **A** is $(\mathbf{A} \cdot \mathbf{B})/A$. When **A** is a unit vector, $\mathbf{A} \cdot \mathbf{B}$ is the component B_A of **B** in the direction of **A**.		$C = A_xB_x + A_yB_y + A_zB_z$	$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$ $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$ $\mathbf{A} \cdot \mathbf{A} = A^2$
$\mathbf{C} = \mathbf{A} \times \mathbf{B}$	Vector product (cross product)	**C** is orthogonal to the plane containing **A** and **B**, is of length $AB\sin\theta$, and points from the plane in the direction of the right thumb when the right hand is parallel to **A** and the fingers curl from **A** to **B**.		$\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$ $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ $\mathbf{A} \times \mathbf{A} = 0$

Note: These operations are defined for arbitrary vectors, but the vectors often must be moved into the required correspondence with each other. In doing so, we move them without altering either their length or their direction.